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Whilst every other branch of Mathematics has made the largest progress from discovery to discovery, ELEMENTARY GEOMETRY alone,—the earliest discovered, and therefore the oldest, theory in Mathematics,—has remained almost stationary for the last two thousand years. There is now, however, a new and strong desire stirring for a larger evolution of this Science, and for a wider development of it upon the old Euclidean lines.

The present work is an attempt to solve the difficult question, which is at present much agitated among teachers and students of this science: *How can Geometry be best presented in a pure scientific form?* This question would have received a practical answer long ere this, but for the tendency natural to human beings to turn teachers into infallible authorities, and great men into deities. The help in geometrical science which Euclid gave, the aid in mental science afforded by Aristotle, became, in the course of ages, hardened into barriers against progress, and hindrances to the development of the scientific spirit in these subjects. But Aristotle is now better understood; and his spirit is better obeyed, even if there is less servile attachment to the letter of him. So likewise in the SCIENCE OF GEOMETRY, the time has now arrived when we can work in the true light of Euclid, and can receive from him the full benefit of his scientific spirit, without being trammelled by his heavier bonds. The application of Logic to *Geometry still remains what* Euclid made it; it is time that, led by him, we should try to develop this application still further, and to follow this development trustingly to the goal whither it leads us.

The present little work, accordingly, is presented to the public as a fuller development of Euclid's matter and Euclid's method, and as embodying a *Complete Exposition of Elementary Geometry.* To keep faithful to him, it has declined to insert into Euclid's Elements the patchwork of modern formulæ, which give to many editions of Euclid a false look of Analytical Geometry, but which are in every way fundamentally opposed to Euclid's method. The present work, like its prototype, the Greek Elements, is founded throughout upon *pure reasoning.*

If Euclid himself could live once more and re-edit his *Elements,* and give to them a form more in harmony with modern science, it is probable that a number of teachers would continue to prefer the accustomed bonds of the old Mummy to the living method of the new Master. But this has not been the feeling of the present writer. The spirit of Euclid has inspired him throughout this Book, if the letter of Euclid has not been closely adhered to. Whatever new developments are to be found in the book are due to Euclid himself and to his method; for they have been gained by following his example, and by immersing the old and well-known *Elements* in the waters of its original spring—that is, LOGIC, from which all its life, old as well as new, has risen.

This volume contains only the PLANE GEOMETRY, divided into FOUR BOOKS:

Book the First: On Lines.
Book the Second: On Angles and Proportional Lines.
Book the Third: On Plane Figures.
Book the Fourth: On Areas.

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GEOMETRY.

THE ELEMENTS
OF
GEOMETRY
IN EIGHT BOOKS
OR
FIRST STEP IN APPLIED LOGIC

BY
L. J. V. GERARD

PART I
PLANE GEOMETRY

BIBLIOTHECA BODLEIANA

> "I believe that a work on the elements of geometry can be made in which all things should be treated in their natural order, all propositions proved in very simple and natural ways, and in which, nevertheless, everything should be most clearly demonstrated."
>
> THE PORT ROYAL LOGIC.

LONDON
LONGMANS, GREEN, READER & DYER
PATERNOSTER ROW
1874

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The Selwood Printing Works,
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PREFACE.

It has been the aim of the author of this book to work out such a system as could be made throughout an illustration of the elementary laws of Logic; simple enough to initiate the young to this important science, and in studying which his mind could make its first practice in wrestling with argument. As yet, no part of mathematics can so easily bear a purely logical treatment as Geometry: its objective part may be readily exhibited in an almost tangible form. For this reason the author has selected Geometry as his subject, as being the one most fitted for the embodiment of the rules of Reasoning; and in doing so, he has followed the example of the ancients.

In order to succeed in the treatment of any science, it is first of all necessary to clear it, not of that which is unconnected with it, for all sciences are more or less connected together, but of all that is *foreign* to it, and not strictly *comprised* within it as a vital part. Accordingly the author has carefully discarded from Geometry all that which properly belongs to Arithmetic; while those parts which relate to measurement, and thus really pertain to both, he has treated separately, delineating as precisely as he could the demarcation between the two sciences. He has only admitted a part of the Theory of Proportions, and has added it as a purely arithmetical digression to the Introduction, for the sole purpose of uniformity.[1] He has also scrupulously refrained from adverting to the applications of Geometry to Sciences, Industries, or Arts, as the mere mention of these numberless applications would have irretrievably obstructed the path of pure Logic, and concealed the true track from the dazzled eye of the student. How can the applications of Geometry to Mechanics, for instance, be shown otherwise than by means of a complete treatise on that science? The same may be said of those two important auxiliaries to Civil Engineering, the arts of Surveying and

---

[1] The question of incommensurable quantities, which has been the subject of such endless discussions, is a purely arithmetical one. The ratio of two geometrical quantities may always be replaced by that of two straight lines, which have at least, according to the author's theory, their infinitely small *elements* as a common measure.

Levelling. Both of them are, so to speak, the most palpable forms in which Geometry is applied in practical life: still, both must needs be treated on their own merits, and quite independently of their theoretical substratum. The subjects of study in Geometry have a characteristic feature, owing to which the boundaries of that science may be defined more easily than those of any other; namely this that, being confined, as it were, to the human mind alone, their existence can only be foreshadowed to the organ of vision by means of graphic delineation, without being actually realized by the senses.

The first task devolving on the author was to determine upon the method best suited to his purpose. In Euclid's Elements, which seem to be a combination of the works of several authors, and a compromise between various diverging theories, the method is multiform, and often untraceable, owing to the very complicated nature of the system. But, in the treatment of any subject, it is essential that the method should ever be visible so as, like Ariadne's thread, to guide the inexperienced student throughout. Thus, it is most important to adopt a method which explains every part, and connects the whole naturally together in one body; and the best method will be that which does so in the most simple and lucid manner, no gap, no jerk, being allowed. It will be said that, for each subject, there is but one real method which can produce a perfect system, and unless that one true method be discovered, all efforts must prove nugatory. Now, perfection is not absolute, but relative; what is perfection to-day, will no longer be so to-morrow: it is a growing thing. The number of geometrical truths increases constantly, and a considerable addition to those known at present may be expected in the future, when the icy spell, under which Geometry is now lying dormant, shall have been broken. Every new discovery tends to exhibit more clearly the real connection, the *Causalnexus*, as the Germans term it, existing throughout the whole array of geometrical truths. Their *ensemble* has been compared to a chain; a tree would be a more suitable simile. Geometers discuss the connection of the boughs with the branches, and of the branches with the trunk; and what is looked upon as an integral part of the trunk is often proved, by a more efficient method, to be but a secondary bough; but, let the method be ever so skilful, it can never reach the root or principle, except through the trunk which must consequently be clearly marked at the outset; and the more clearly it exhibits the whole system,—root, trunk, branches, and boughs,—their connection and their dependence on one another,—the nearer it comes to Nature itself.

A strictly logical system of Geometry has for each proposition a fixed place, which it is the task of the geometer to discover, and which cannot be altered either to facilitate a demonstration, or for any other purpose. The frequent deviation from this rule seems to be the chief defect of Euclid's Elements; and this drawback must exist in all systems which are the result of a compromise. Upon the logical plan, the propositions must be arranged, according to the laws of the human mind, synthetically, from the simplest to the most complex, or analytically, from the complex to the simple, but always naturally, without disproportion or distortion of any kind.

The author refrains from going at full length into the principles on which his system is founded, because, on the one hand, these will be readily understood by a perusal of his work; and on the other, because the present treatise is designed but as a transition stage between the Euclidean plan and the author's ideal of a system of Geometry, which, he fears, would have been thought prematurely novel to-day, but which he still hopes to publish at some future time; and again, because he feels that his labours, if successful, will be only preparatory to the great crisis which must soon settle Geometry as a science, when the time and the man are come. It is enough that he should now state that, however much he has departed from Euclid in the general arrangement of The Elements, and in many points of detail, he has still endeavoured to preserve throughout the main characteristics of Euclid's method. He has striven to attain that unity and plasticity that are wanting in Euclid's system, and which are indispensable to its further development.[1] Thus, although he has altered almost every one of Euclid's definitions, he has but slightly modified his definition of the straight line; and this very definition thus modified, he has made the cornerstone of his whole system. Above all, he has tried to preserve the extreme rigour of reasoning, the admirable accuracy of thought and expression, which are the immortal glory of the great Geometer; he has adhered to the spirit of Euclid's fabric, if not to its letter.

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[1] The often repeated praise given to Euclid, that his book is, after two thousand years, still used by teachers in the exact form he has written it, must, the author thinks, hardly please his shade, as such praise is almost equivalent to the reproach of having stopped short the development of Elementary Geometry: the proof of life is *progress.*

In writing the present work, the author has been in a great measure prompted by the desire of supplying a really elementary text-book by which "the Royal Road to Geometry" might be opened up to young students of both sexes; so elementary indeed as to place it within the reach of even children, as a means of teaching them to think. The author refers deliberately to both sexes, for, in his humble opinion, woman stands even in greater need than man of mathematical training as a practical course of logic, to compensate the great excitability of her nervous system, and her inherent propensity to yield to feeling and passion rather than to strict reasoning. Hitherto, mathematics in general, and Geometry in particular, have been a sealed book for "the million"; it is only of late years that the tendency to keep the *profanum vulgus* at a respectful distance, has somewhat subsided among geometers. The author, so far from wishing to keep the masses in ignorance, would rather throw the doors of Geometry wide open to them, and make this science part and parcel of elementary instruction for all classes of society. Let that veil be removed, which can no longer have any other object but to "wrench awe from fools, and *tie the wiser* souls by its false seeming";

let "the case and habit of place and form," to use Shakspeare's own words, become generally known, and a vast stride will have been made towards that popularisation of knowledge which our age is now endeavouring to achieve.

But the author is not blind to the fact that rendering Geometry, and indeed all sciences, easier to learn, will not improve greatly the instruction given in schools, unless the teachers simplify it still further by a rational and judicious method of teaching. In using this book, the teacher who means to do really efficient work, will have to explain thoroughly the definitions contained in each chapter, illustrating each point with numerous diagrams, before the theorems are read; he will have to supply, at every stage of the demonstration of each proposition, supplementary diagrams for the complete mastering of each difficulty. Whenever it is possible, he should give other demonstrations than those contained in the book, and make his pupils in their turn try to form others for themselves, never omitting to discuss their merits and demerits at full length. As every-day exercises, he should require from them the demonstration of each corollary, or else give it himself; for, on no account must any theorem be taken up before the demonstration of all the corollaries of the preceding one be gone through. Also, he should constantly test the reasoning and inventive powers of his pupils, by proposing them new theorems to be demonstrated, and problems to be solved and discussed, by means of the theorems and corollaries previously read. And last, not least, he should assist the students in finding out the whole of the inverses and converses of each proposition, and prove their truth or falseness as the case may be. In a word, the teacher must apply himself to the spirit, and not merely to the letter of the book. The way in which a text-book of Geometry should be used is forcibly expressed in the following passage from a speech of Dr. Hirst, President of the *Association for the Improvement of Geometrical Teaching*:—

"The text-book indeed, to be properly used, should always be subordinated to the teaching; but to do this it is absolutely essential that the teacher should, by his own study, have risen not merely up to, but above the level of the text-book he employs. Until he has so mastered the subject that it has become plastic in his hands, his teaching must necessarily remain defective; for geometrical truth, it must be remembered, has, like all other truth, many sides, and no text-book can present all or necessarily *the one which,* to individual pupils, is the most accessible. Alternative methods of demonstration, *inquiries into the interdependence of propositions,* judicious variation of data, and just discrimination between the contingent and necessary properties of figures;—these and numerous other matters, all essential to geometrical culture, can only be properly supplied by the teacher; no text-book could be weighted with them. Above all it is to him that we must mainly look for the cultivation of that scientific method of inquiry under whose guidance solely problem-solving can be raised in character above what has been termed 'exalted conundrum guessing,' and acquire its full educational value."

The author takes this opportunity to acknowledge the great help he has derived from J. M. C. DUHAMEL's book: *On Methods in Argumentative Sciences,* and to express his deep regret at not having been able to avail himself of J. DELBŒUF's valuable treatise: *Philosophic Prolegomena of Geometry,* and of J. UEBERWEG's dissertation: *Scientific Exposition of the Principles of Geometry,* at an earlier stage of his labours; these last two works not having come to his hands until one half of The Elements had already gone through the press.

In conclusion, the author begs to dedicate his work, which is the result of long persistent labours, to all

Teachers and Students who appreciate the sincerity of his intentions, and who sympathize with his efforts. The book lays no claim to perfection. Many perhaps will find numerous defects in it, whilst the most benevolent of readers will probably hold that it is open to considerable improvement. To all the author begs to say that he courts criticism in the interest both of his work and of the cause it is designed to subserve; and he will look upon all those as personal friends who, by pointing out to him such imperfections and shortcomings as they may discover in his book, shall afford him the means of improving it in the future.

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FOR	READ
Which W. T. B. D., or W. W. T. B. D.	Which was to be demonstrated, or Which was to be done.
[*Hyp.*]	By hypothesis.
[*Const.*]	By construction.
[*Dem.*]	By demonstration.
[1]	By Theorem 1 in the same Book.
[I. 1 i]	By Corollary I of Theorem 1 in Book I.
[i]	By General Theorem I, page 16.
[ix]	By Arithmetical Theorem IX, page 24.
[*Post.* I]	By Postulate I.
[*Def.* 1]	By Definition 1 in the same Book.
[I. *Def.* 1]	By Definition 1 in Book I.
[*Gen. Def.* 1]	By General Definition 1, page 1.
[*Probl.* 1]	See Problem 1 at the end of the same Book.
[I. *Probl.* 1]	See Problem 1 at the end of Book I.
(I. 1 i)	See Corollary I of Theorem 1 in Book I.
(I. *Def.* 1)	See Definition 1 in Book I.
(*Eucl.* I. 1)	See Euclid's Elements, Book I, Proposition 1.
(Conv. of 1)	Converse of Theorem 1.
+	Plus.

THE ELEMENTS OF GEOMETRY.

INTRODUCTION.

I. GENERAL DEFINITIONS.

> "To the geometers alone belongs the advantage of having banished from their schools and books controversy and dispute." THE PORT ROYAL LOGIC.

1. GEOMETRY is the science of space. Its special object is the *study of the properties* of form and the *measure* of extension.

It admits first of all the existence of space and of its two properties, infinite *extension* and absolute *homogeneity*, and presupposes the idea of *form.*

Form is not a property of space, but is a property of all limitations, or rather determinations, of space. Space is determined by every object in the universe; our mind also conceives in space numberless different determinations which may, or may not, exist in nature.

ON EXTENDED SPACE.

2. *Geometrical figures,* or simply *figures,* are mental determinations of space.

Space is of itself indeterminate and infinite[1]; the determinations which our mind creates in space are alone within the scope of Geometry. Geometrical figures exist in our imagination only, but are more or less realized in nature and art.

---

[1] Throughout The Elements the meaning of the word *infinite* will be restricted to that which it is impossible to realize in imagination. The word *indefinite* will be used to mean *indeterminate with regard to extension* only.

3. The *place* of a figure is the portion of space which it occupies; figures which have the same place are said to *coincide* with each other.

Or, the place of a figure is the portion of space in which we conceive it to exist. As geometrical figures are mere conceptions of our mind, they may penetrate each other, and partly, or wholly, coincide with each other; also, we may at will increase or diminish their extension, or change their form.

Geometrical figures are of four kinds:—*solids*, *surfaces*, *lines*, and *points*.

4. A *solid,* or *geometrical body,* is a limited portion of space. The extension of a solid is its *solidity* or *volume.*

When a portion of space is conceived to be separated from the rest, that by which space is conceived to be divided is called *surface*. Whence

5. A *surface,* or *superficies,* is the limit of a solid. The extension of a surface is its *area.*

Hence, surfaces divide space into determinate portions. A solid, being a portion of space, can be divided by any surface.

6. A *line* is the limit of a surface. The extension of a line is its *length.*

Hence, lines divide a surface into determinate portions, and the intersection of two surfaces is a line. By abstraction, one may have the idea of a line existing in space independently of any surface.

7. A *continuous* line is that which has no extremities.

8. The *distance* between two figures, or two parts of a figure, is the length of the shortest line which may unite them.

9. A *point* is the limit of a line.

Hence, points divide a line into determinate portions, and the intersection of two lines is a point. By abstraction, one may have the idea of a point independently of any line: that is, a point may exist anywhere in space, as well as anywhere in the whole extension of a line or surface. It is obvious that a point has no extension, and is limited by itself. A point is, then, the opposite of space: the latter being infinitely great, whilst the former is infinitely small.

The definitions of the surface, line, and point, may be summed up thus: a *surface* is any section, or division, imagined in space; a *line*, any section imagined in a surface; and a *point*, any section imagined in a line.

By means of motion, a solid, a surface, or a line may be generated by its limit. Thus, if a point is imagined to move out of itself, this will raise in the mind the idea of length, and produce a line; the motion of a line out of its length will generate a surface; and the motion of a surface out of its own extension will produce a solid. It may be seen thus that a solid contains points, lines, and surfaces, and may even contain other solids; a surface contains points and lines, and may also contain other surfaces; a line contains points, and may contain other lines; but a point has no parts.

10. The *direction* of a line is its position relatively to each of its points.

In the extension of a line there is an infinity of points, for between two points, however close to each other, it is always possible to conceive a third one. But it is in the very nature of the point that, when two points touch each other, they coincide with each other, and become indivisible; therefore, if all the points of a line should touch each other, the whole line would be reduced to a point: there must be consequently some distance between the points of a line, however small that distance may be.

11. The *elements* of a line are the infinitely small parts thereof, which unite its points.

The extent of the motion necessary to a figure for producing the figure which it limits is called a *dimension* of the latter. Hence, *space has three dimensions.*

These three dimensions are called *length*, *breadth* (or *width*), and *height* (or *depth* or *thickness*). It is evident, from their definition, that they may be taken indifferently one for the other.

Of the three dimensions of space, a point has none; a line has one, that is, length; a surface, two, length and breadth; a solid possesses the three dimensions; whence it is seen that the three dimensions are the elements, or constituent parts, of extension.

The *extension of a solid has the* three elements, that of a surface has two, and that of a line has only one: so that one of the three elements of extension, length, may be represented by a line; and, as the three dimensions may be interchanged, or taken indifferently one for another, it follows that *they may be considered as so many lines.*

ON HOMOGENEOUS SPACE.

12. A *geometrical magnitude,* or simply *magnitude,* is the extension of a geometrical figure.

Thus there are three kinds of magnitudes: *lengths, areas,* and *volumes.*

Space being homogeneous, it follows that there is no absolute geometrical magnitude, and that a figure is great only relatively to another.

13. A magnitude that can be contained an exact number of times in another is said to be a *measure* of it.

The only means of conceiving an exact idea of a magnitude is to *measure* it. All that is related to measurement itself is studied in Geometry, but the speculations about the results of measurement, that is, about numbers, form the science of *Arithmetic.*

14. *Measuring* a magnitude is comparing it with another of the same kind, well defined and supposed to be for the moment absolute, to which is given the name of *unit.*

The mind not being able to compare any but such magnitudes as have properties in common, it is obvious that the extension of any figure, that is, its magnitude, can only be measured by the magnitude of a figure of the same kind. Thus the unit for measuring solidity, must be a volume; the unit of area must be an area; and that of length, a length. But, as the elements of extension may be considered as so many lines, *all measurement in Geometry is reduced to the measuring of lines.*

Two magnitudes of the same kind may be compared to ascertain either by how much the one is greater than the other, or how many times the one is contained in the other: the result of the comparison is called in the first case *difference,* in the second, *ratio.* The study of differences and ratios belongs to the province of Arithmetic, and is not dealt with in Geometry save in connexion with form, for the completion of the theory of similarity and to determine more accurately the boundaries between Geometry and *Arithmetic.*

15. The *value* of a magnitude is the number expressing how many times it contains the unit.

That is, the value of a magnitude is its ratio to the unit. It is evident that the unit used to measure any magnitude is arbitrary, and may always be changed into another; and also that any magnitude may be chosen as a unit for measuring all magnitudes of the same kind.[1] It is also evident that the magnitude of one figure cannot measure that of another with which it cannot partly coincide; and, again that the magnitudes of figures which cannot partly coincide, have no common measure, and consequently no common unit.

16. *Incommensurable* lines are lines which have no other common measure than their infinitely small elements.

With regard to the properties of figures, Geometry generally leaves their magnitude out of consideration; it only studies that in its relations with form. Magnitudes, abstraction being made of the figures of which they are the extension, take in General Mathematics the name of *quantities*; their study is the province of *Arithmetic*.[2]

In Arithmetic, form is indeterminate; the magnitude may be that of a figure of *any* form. In Geometry, on the contrary, it is magnitude which is indeterminate; a figure may be of any magnitude, even infinitely great or infinitely small: in the latter case, it is reduced to a point.

If, in a solid, one of the dimensions be conceived to diminish until it becomes *nought*, the solid, having then only two dimensions, will become a surface; thus a surface is a solid the thickness of which is nought, that is, infinitely small. Also, a line may be considered as a surface the breadth of which is infinitely small, or as a solid the thickness and breadth of which are infinitely small. Again, a point is a line, the length of which is infinitely small, or a surface the length and breadth of which are infinitely small, or a solid the length, breadth, and thickness of which are infinitely small: in short, a point is a figure of any kind, the dimensions of which are infinitely small.

The idea of *infinitely small* is thus produced by denying magnitude to any figure; in the same way, the idea of *infinitely great* is acquired by denying form to a figure. In denying the idea of a terminal point to a line, the mind conceives a line infinitely great; also, an infinite surface is that to which the idea of a terminal line is denied; and an infinite solid is that to which the idea of a terminal surface is denied.

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[1] It is desirable that men of all nations should agree on using the same unit for each kind of magnitude, and that this unit should be the simplest and best known of all.

[2] Therefore the whole theory of ratios and proportions is a part of Arithmetic and is supposed to be known by the student of Geometry. By Arithmetic is meant here also *generalised Arithmetic or Algebra.*

Inasmuch as, according to the preceding, the extension of a point added to any line, or the breadth of a line added to any surface, or the thickness of a surface added to any solid, does not alter the magnitude of the line, the surface or the solid respectively—the number of points contained in a line, or of lines contained in a surface, or of surfaces contained in a solid, is infinite: it is evident that a quantity infinitely small repeated a finite number of times cannot produce a finite quantity.

It is obvious, also, that an infinite line cannot be contained in a finite surface or solid, nor an infinite surface or solid in a finite solid.

From what has just been said it will be admitted that the magnitude of a geometrical figure is essentially distinct from its form, since magnitude is common to all figures, and is consequently immaterial in the study of an isolated figure.

17. The description of the form, magnitude, and place, of a particular figure is called its *construction*.

If, in the construction of a figure, the magnitude be not stated, then the form only is determinate, and the figure may be of any magnitude.

From the homogeneity of space it follows that a determinate portion, and in fact any portion of it, may differ from another of the same form by its magnitude, but not by its properties.

Several figures may therefore have the same form and the same magnitude, or the same form only, or the same magnitude only.

18. Figures which have the same form and the same magnitude, are *equal to each other*, or simply *equal*. (*Eucl. Ax.* 8.)

That is, figures are equal when they may be taken indiscriminately one for the other.

From the homogeneity of space, it follows immediately that two or more equal figures may coincide in all their parts, and also that equal figures which coincide in all their parts, form but one figure and are not distinct from one another: they have consequently the same properties.[1] It is therefore evident that, in the study of Geometry, any figure may always be replaced by its equal, and also that if two or more figures be proved to coincide in all their parts, they are equal. Again, two figures, each of which is equal to a third, are obviously equal to each other, for they may both be made to coincide with the third, and consequently with each other.

19. Figures which have the same form, but whose magnitude is indeterminate, are *similar to each other*, or simply *similar*.

Similar figures which have the same magnitude are necessarily equal to one another; and two figures, each of which is similar to a third, are similar to each other, for they have all the same form.

20. Figures which are equal, or similar, each to each, to the same number of other figures placed in the same order, are said to be placed *similarly* to the latter.

21. Figures which have the same magnitude, but which may have any form, are *equivalent to each other*, or simply *equivalent*.

Equivalent figures have obviously the same value, and conversely, figures which have the same value are equivalent to each other; also, it is evident that there are no equal magnitudes but those of equivalent figures, and that, when geometrical figures are compared to one another in point of magnitude, equal magnitudes may be interchanged with each other, although they may not belong to equal figures.

Two equivalent similar figures are equal to each other; also two equal figures are necessarily similar and equivalent, but two similar, or equivalent, figures need not be equal.

In accordance with the preceding definitions, it is always possible to change the magnitude of a figure and yet preserve its form, and to change the form of a figure without changing its magnitude.

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[1] When two or more figures are spoken of, distinct figures only are meant, unless the contrary is expressly *stated.*

22. *Expanding* a figure is increasing its magnitude without altering its form; *reducing* a figure is diminishing its magnitude without altering its form. *Reducing* two figures *to the same magnitude* is reducing the greater, or expanding the lesser, of them until they become equivalent.[1]

Two figures are necessarily similar to each other if, by reduction, they become equal to each other; also, it is evident that, if two similar figures be reduced to the same magnitude, they will be equal to each other.

23. *Transforming* a figure is changing its form without altering its magnitude.

Geometry, like other sciences, has necessarily a descriptive part, which consists in the definition and classification of figures and their properties; the science itself studies these properties by means of Reasoning.[2]

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[1] For the sake of brevity, the term *reducing* is used in The Elements for both *expanding* and *reducing*: that is, *altering the magnitude of a figure without altering its form*; unless the sense indicates clearly that *expanding* is not meant.

[2] The short digression on Reasoning which follows forms no part of Geometry, but is given here as a help to the student who has not been previously acquainted with Logic. The terms not defined here are those used by logicians, and are supposed to be known.

II. ON REASONING.

The principle of geometrical reasoning is *deduction*, and the method for the whole science is *synthesis*.

Although the general method of Geometry is synthetical, geometers often make use of *analysis* in particular cases, as a help to synthesis.

Every new thing spoken of in a science must have some relation to a thing already known; the clear statement of such relation, made to distinguish the new thing from any other, is called a *definition*.

The enunciation of a truth, or of what is supposed to be true, is called a *proposition*.

Two propositions are said to be *contradictory to each other*, or simply *contradictory*, when the one affirms what the other denies; so that if the first be true, the second is false, and, *vice versâ*, if the first be false, the second is true.

An *axiom* is a self-evident truth.

Axioms are the foundation of reasoning; the following only are used in Geometry:—

1. *The whole is greater than its part.* (Eucl. Ax. 9.)
2. *The whole is equal to all its parts taken together.*

The name of *postulate*, or *demand*, is given to truths less evident than axioms, but the admission of which may be demanded without proof.

As a consequence of the homogeneity of space, let the following general postulate be granted:—

I. *In the reduction of a figure, every part thereof is reduced.*

A *theorem* is a truth which is not evident of itself, but which becomes so by reasoning.

The enunciation of every theorem is composed of two parts: the *hypothesis,* which is the supposition made on the subject of the theorem, and on which the reasoning must be founded; and the *conclusion,* which is the inference deduced from the hypothesis, and must be the aim of the reasoning. The hypothesis and conclusion are said to be *complex* when there is more than one supposition made on the subject, or more than one inference drawn from the hypothesis.

The *inverse theorem,* or simply *inverse* of a theorem expresses that if the conclusion does not exist as true, the hypothesis does not exist either; the truth or falseness of the inverse of a theorem is directly deduced from that of the theorem itself. When the hypothesis, or the conclusion, is complex, there are as many inverses to the theorem as there are suppositions made on the subject or inferences drawn from the hypothesis.

The name of *converse theorem,* or simply *converse* of a theorem, is given to a new theorem in which the conclusion is taken as a hypothesis, and the hypothesis as a conclusion. There are as many converses to a theorem as there are inverses. When the converse of a theorem expresses a fact more general than that which is expressed by the theorem itself, it often happens that the converse is false; in which case its inverse is equally false.

The *opposite theorem,* or simply *opposite* of a theorem, is the inverse of its converse. When the opposite of a theorem already proved, is proved also, the theorem acquires, if possible, a greater appearance of truth.

The reasoning by which the truth of a theorem is proved is called *demonstration.*

There are several kinds of demonstrations :—

The *direct,* or *synthetical,* demonstration consists in stating propositions admitted to be true, deducing others as necessary consequences of them, then new ones from these, and so on until the proposed theorem is reached, and shown to be equally true. The only difficulty is to choose the starting propositions, and to choose among their consequences those leading to the proposed theorem.

The demonstration *by superposition* is a direct demonstration which proves the equality or inequality of geometrical magnitudes by applying them to one another to make them coincide; it follows naturally from the definition of geometrical equality.

The *indirect*, or *analytical*, demonstration, specially used for finding a direct demonstration, consists in showing that the proposed theorem is a necessary consequence of another proposition; the latter, the necessary consequence of another; and so on until a proposition is reached which is already admitted to be true. The difficulty is to choose, among the propositions of which the proposed one is a necessary consequence, one which leads to a proposition already admitted as true, and above all to choose one which does so by the shortest possible way.

The *mixed* demonstration uses synthesis and analysis alternately.

The *arithmetical* demonstration is a direct, or indirect, or more often mixed, demonstration, which establishes the truth of a theorem by arithmetical operations on the magnitudes of the hypothesis.

The demonstration *by impossibility*, or *by absurdity*, is a special kind of demonstration founded on the principle that the truth of a proposition proves the falseness of its contradictory, and conversely. This kind of demonstration, which, if it convinces the mind, does not always enlighten it, must be used sparingly and with much caution; for the mind is not satisfied unless it knows, not only that a thing is true, but also why it is so. It ought to be used only when the truth to be proved is already almost evident of itself, as is the case with the inverses of a theorem already demonstrated; apart from such cases, it is allowable only where no better method can be found.

The general rules of Reasoning are three :—

1. To define every term clearly, so that there may be no ambiguity in its meaning.

2. To prove all assertions by means only of *definitions* previously laid down; of *principles* granted as self-evident, or which are so clear that they cannot be contested; or of *assertions* previously proved, and which, by reasoning, have become equal to principles.

3. To prove nothing which requires no proof.

It is very important in demonstrations to prevent all ambiguity by *always substituting mentally for each term the definition which explains and limits its meaning.*

It is equally important to follow the true order of nature, which proceeds from the simple to the complex, and from the general to the particular, and to avoid proving simple notions by means of complicated ones, as well as using any far-fetched methods.[1]

A *lemma* is a preparatory theorem used to facilitate the demonstration of a more important one.

A *corollary* is an accessory truth which follows immediately from the demonstration of a theorem. The distinctive feature of a corollary is that it requires no demonstration, or only a very simple one.

If it be proposed to construct a figure upon certain known conditions, the enunciation of these conditions and of the figure to be constructed, is called a *problem;* the known or given conditions are termed *data.*

The indication of the different stages of the construction required in a problem, is called the *solution* of the problem. It is evident that according to the number of figures which may satisfy the data, a problem may have one solution or more, or that there may be none at all: in the latter event, the problem is said to be *impossible.* A problem which may have an infinite number of solutions is called an *indeterminate* problem.

A solution, to be complete, must be proved by a demonstration to satisfy the data of the problem; in the case of no solution, it must be proved that none can satisfy them.

The *discussion* of a problem is the determination of the several solutions resulting from all possible suppositions made on the data.

There are two kinds of solutions to a problem.

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[1] Let the student find for each theorem other demonstrations than that given in The Elements, and discuss the merits of each of them.

The *direct,* or *synthetical,* solution substitutes for the proposed problem a simpler one, the solution of which is known or admitted, deducing from it the solution of another, then deducing from the latter the solution of a third one, and so on, until the proposed problem is reached and solved. The difficulty is to choose the starting problem, and, among the problems the solutions of which are deduced from a previous one, to choose one leading to the proposed problem.

The *indirect,* or *analytical,* solution substitutes for the proposed problem a simpler one, of whose solution that of the proposed problem is a necessary consequence; replacing then, in the same manner, the second problem by one simpler still, and so on, until one is reached the solution of which is known or admitted. The only difficulty is to choose, each time, a problem really simpler than the preceding one, and, above all, one which leads by the shortest possible way to a problem already solved.

For communicating to others the known solution of a problem, the direct, or synthetical, process is preferable; but the student who wants to find out the solution of a proposed problem cannot use it, for he would not know what solved problem to start from. He might use it, if he wanted to find solutions for new problems not yet proposed to him; but when the problem is proposed, the indirect, or analytical, process is the only one to which he can properly resort to find the required solution, because he may see the relation between that solution and the simpler problem which he substitutes for the proposed problem, whilst it is often difficult for him to see in what relation the latter may stand to any problem of which he already knows the solution.

A *scholium* is a mere remark made to show the connection of some propositions with one another, or to point out their utility.

III. ON THE REPRESENTATION OF GEOMETRICAL FIGURES.

As a help to reasoning, the demonstration of a theorem, or the solution of a problem, generally requires a drawing representing the figure which is the subject of the reasoning.

The drawing representing a geometrical figure is called a *diagram*, or *figure;* the principal points of it are marked with letters of reference.

A point is marked with one letter only; a line is indicated by the letters placed at its principal points. A surface is indicated by the letters of its principal lines, and a solid by those of its principal surfaces. Sometimes one letter indicates a whole line, surface, or solid.

The letters generally used are *Roman* or *Italic capitals;* but sometimes small letters, or Arabic and Roman figures, and even Greek letters, are used. Dashed and numbered letters are used in special cases.[1]

The symbols of notation are the same as those used in Arithmetic.

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[1] The student must form the habit of changing the letters used in the diagrams, and the diagrams themselves; he must also accustom himself to demonstrating without the use of any diagram.

IV. GENERAL THEOREMS ON QUANTITIES.

To facilitate the comparison of magnitudes in geometrical propositions, some theorems on quantities will have to be demonstrated here which properly belong to General Mathematics. These theorems and their corollaries may also be accepted without demonstration, and regarded as *General Postulates.*

DEFINITIONS.

24. The *sum* of several quantities is another quantity equal to all of them together; the sum of two equal quantities is the *double* of each of them; the sum of three equal quantities is the *triple* of each of them. *Adding* several quantities to each other is determining their sum.

Evidently, the sum of several quantities contains as many times their common unit as the unit is contained in all of them together.

25. *Multiplying* a quantity by a number is determining the sum of the same number of quantities, each equal to the first; the sum is the *product* of the first quantity by the number.

Thus the double, or triple, of a quantity is the product of that quantity by two or three.

26. The *difference* of two quantities is a third quantity equal to the excess of the one over the other; if the difference of two quantities be equal to one of them, the latter is the *half* of the other. *Subtracting* a quantity from another is determining their difference.

Evidently the difference of two quantities contains as many times their common unit as the unit is contained more in one than in the other. It is also evident that if the difference of two quantities be added to the lesser, the sum will be equal to the greater.

27. *Dividing* a quantity by a number is dividing the quantity into the same number of equal parts: each part is the *quotient* of the quantity by the number.

It is obvious that if the quotient of a quantity by any number be multiplied by the same number, the product will be equal to the first quantity.

POSTULATE II.

The sum, or difference, of two quantities is not altered when one of them is replaced by an equal quantity.

THEOREM I.

(*Eucl. Ax.* 1.)

Two quantities, each of which is equal to a third, are equal to one another.

Let there be three quantities, called respectively A, B, C, and let A and B be each equal to C.

The quantity A is equal to C [*Hyp.*], and the quantity C may be replaced by its equal B [*Post. II*]; then A is equal to B: but the same reasoning could be applied to any three quantities other than A, B, and C, two of which are each equal to the third; therefore any two quantities equal to a third are equal to one another, which was to be demonstrated. Inversely

If, of two unequal quantities, one is equal to a third, the other is not.

For, if they were both equal to the third, they would be equal to one another, which is impossible [*Hyp.*].

COROLLARY. *Two quantities, each of which is equal to equal quantities, are equal to one another.*

THEOREM II.

(*Eucl. Ax.* 2 *and* 3.)

If the same quantity be added to, or subtracted from, each of two equal quantities, the results shall be equal to each other.

Let there be three quantities A, B and C, and let A be equal to B.

The quantities A and C together are evidently equal to A and C: but the quantity A may be replaced by its equal B [*Post.* II]; then A and C are together equal to B and C. The same reasoning would prove also that the difference of A and C is equal to that of B and C; therefore the sum, or difference, of A and C is equal to that of B and C, which W. T. B. D. Inversely

If, after adding the same quantity to, or subtracting it from, two other quantities, the results be unequal, the two last quantities are not equal to each other.

COROLLARY. *If the same quantity be added to, or subtracted from, each of any number of equal quantities, the results shall be equal to one another.*

THEOREM III.

(*Eucl. Ax.* 4 *and* 5.)

If the same quantity be added to, or subtracted from, each of two unequal quantities, the results shall be unequal.

Let there be three quantities, A, B and C, and let A be greater than B.

Let D be the difference of A and B; then A is equal to the sum of B and D. Because the sum of B and D is equal to A, the sum of B, D and C is equal to that of A and C [ii]: but the sum of B, D and C is evidently greater than that of B and C; therefore the sum of A and C is greater than that of B and C.

Let B be greater than C; then A is also greater than C. Because A is equal to the sum of B and D, the difference of A and C is equal to that of $B + D$ and C [ii]: but the difference of $B + D$ and C is evidently greater than that of B and C; therefore the difference of A and C is greater than that of B and C.

Therefore the sum, or difference, of A and C is not equal to that of B and C, which W. T. B. D. Inversely

If, after adding the same quantity to, or subtracting it from, two *other quantities, the results be equal, the two last quantities are equal to each other.*

Corollary. *If a quantity be greater than another and the same quantity be added to, or subtracted from, each of them, the first sum, or difference, shall be greater than the second.*

Theorem IV.

(*Eucl. Ax.* 2 *and* 3.)

If two quantities are equal to two others, each to each, the sum and difference of the first two quantities are respectively equal to the sum and difference of the other two.

Let there be four quantities, A, B, M, N; let A be equal to B and M equal to N.

Because A is equal to B, the sum, or difference, of A and N is equal to that of B and N [ii]: but N is equal to M [*Hyp.*]; then the sum, or difference, of A and M is equal to that of A and N [*Post.* II]; therefore the sum, or difference, of A and M is equal to that of B and and N [i], which W. T. B. D. Inversely

If the sum, or difference, of a group of two quantities be unequal to the sum, or difference, of a group of two others, and if one quantity of the first group be equal to one of the second, the other quantities are unequal.

Corollary. *If any number of quantities are equal to the same number of other quantities, each to each, the sum of the former is equal to the sum of the latter.*

Theorem V.

If two quantities are respectively greater than two others, the sum of the former is greater than that of the latter.

Let there be four quantities A, B, M and N; let A be greater than B, and let M be greater than N.

Because A is greater than B, the quantities A and N are together greater than B and N [iii]; because M is greater than N, the quantities A and M are together greater than A and N [iii]: therefore A and M are together still greater than B and N, which W. T. B. D.

COROLLARY. *If any number of quantities are respectively greater than the same number of other quantities, the sum of the former is greater than that of the latter.*

THEOREM VI.

(*Eucl. Ax.* 6.)

The products of two equal quantities by the same number are equal to each other.

Let there be two equal quantities A and B.

Because A is equal to B, the quantities A, A, A, A and A are respectively equal to B, B, B, B and B; thence the sum of *five* times A is equal to the sum of *five* times B [iv]: therefore the product of A by *five* is equal to that of B by *five*. The same reasoning could be applied to any other number than *five*; therefore the product of A by any number is equal to that of B by the same number, W. W. T. B. D.

Inversely

If the products of two quantities by the same number be unequal, these quantities are not equal to each other.

COROLLARY. *The products of several equal quantities by the same number are equal to one another.*

Scholium. Quantities which are double of equal quantities are equal to one another.

THEOREM VII.

The product of two unequal quantities by the same number are unequal.

Let there be two quantities A and B, and let A be greater than B.

Because A is greater than B, the quantities A, A, A, A and A are respectively greater than B, B, B, B and B; thence the sum of *five* times A is greater than *five times* B [v]: therefore the product of A *by five is greater than that* of B by *five*. The same reasoning could be applied to any other number than *five;* therefore the product of A by any number is greater than that of B by the same number, W. W. T. B. D. Inversely

If the products of two quantities by the same number be equal, these quantities are equal to each other.

COROLLARY I. *If a quantity be greater than another, the product, or quotient, of the first by any number shall be greater than the product, or quotient, of the second by the same number.*

COROLLARY II. *The quotients of equal quantities by the same number are equal to each other,* and conversely.

Scholium. The halves of equal quantities are equal to one another.

THEOREM VIII.

If two equal quantities have a part in common, the parts which are not in common are equal to one another.

Let there be two equal quantities A and B, having a part C in common.

Let D and E be the parts of A and B respectively which are not in common; then D is the difference of A and C, and E is the difference of B and C. Because A is equal to B, the difference of A and C is equal to that of B and C [ii]; therefore D is equal to E, which W. T. B. D. Inversely

Two quantities are unequal if, having a part in common, the parts which are not common be unequal.

COROLLARY. Conversely, *two quantities are equal if, having a part in common, the parts which are not common be equal.*

V. ARITHMETICAL THEOREMS.

Elementary Geometry studies the properties of figures and the measure of their magnitude by means of reasoning only; *Analytical* Geometry is that part of Mathematics which studies the comparison and measure of magnitudes by means of Generalised Arithmetic, or *Algebra*. The following arithmetical theorems are here given to facilitate the understanding of those parts of Elementary Geometry which treat of the magnitude of figures, and are consequently on the boundaries of Elementary and Analytical Geometry.

DEFINITIONS.

28. The number that a quantity is *to* another of the same kind is the number of times, whole or fractional, that the first contains the second, or is contained in it.[1]

Thus, if a quantity contains another as many times as a third one contains a fourth and as many times as a fifth contains a sixth, or if the first is contained in the second as many times as a third one is contained in a fourth, and as many times as a fifth one is contained in a sixth, then the first is to the second as the third is to the fourth and as the fifth is to the sixth.

29. The number that a quantity is to another is called the *ratio* of the first to the second, or simply their *ratio:* the two quantities are the *terms* of the ratio; the first one is called *antecedent*, and the second, *consequent*. A ratio is *reversed* when its terms are interchanged.

It is evident that if the two terms be equal to each other, the ratio is equal to *one*, and conversely, if the ratio of two quantities be *one*, they are equal to each other; also, if the antecedent be greater, or smaller, than the consequent, the ratio is greater or smaller than *one* and conversely.

It is equally evident that two ratios, each of which is equal to a third, are equal to each other [i]; and also that the ratios of two equal quantities to a third one, are equal to each other [i]. Again, two equal ratios remain obviously equal to each other when both are reversed.

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[1] Or sometimes the number by which the one exceeds the other. In The Elements the expression will be used only in the sense expressed in the text.

30. Several quantities are *proportional* when the first one is to the second as the third is to the fourth, as the fifth is to the sixth, and so on: they are then called *proportionals.* Each of four proportional quantities is said to be a *fourth proportional* to the three others.

That is, several quantities are proportional when the ratio of the first group of two is equal to the ratio of the succeeding ones. Hence, if two quantities of the same kind contain respectively as many equal parts as two other quantities of the same, or another, kind, the four quantities are proportional.

31. The expression of the proportionality of four quantities is called a *proportion:* the four quantities are the *terms* of the proportion; the first and the fourth are called the *extremes,* and the others, the *means.* A *proportional series* is the expression of the proportionality of more than four quantities. The terms of the same ratio in a proportion, or proportional series, are *corresponding* terms.

Or, a proportion is the expression of the equality of two ratios; a proportional series is the expression of the equality of more than two ratios. It is obvious that the same mathematical operation which will deduce the first consequent from the first antecedent will also deduce the other consequents from the corresponding antecedents.

32. The antecedents of a proportion are said to be *homologous* to each other, and so are the consequents. The *homologous* terms of several proportions are those which occupy the same rank in each proportion.

33. Four quantities are *reciprocally* proportional when the first two are the extremes, and the last two the means, of a proportion; the four quantities are then called *reciprocal* proportionals.

34. A *continual* proportion is one of which the means are equal to each other: each mean is then called a *mean proportional* to the two extremes, and each extreme is called a *third proportional* to the other terms. Three quantities of which the second is a mean proportional to the two others are said to be *proportional;* they are then called *continual* proportionals.

35. A *multiple* of a quantity is the product of that quantity by a whole number; a *submultiple* of a quantity is the quotient thereof by a whole number.

Hence, if one quantity is a multiple of another, the second is a submultiple, or a measure [*Gen. Def.* 13] of the first.

36. *Equimultiples* of several quantities are the products of these quantities by the same number.

37. The *product* of two quantities, or of one of them by the other, is the product of the first by the number of times that the second contains their common unit; the *product* of three quantities is the product of that product by the third.

38. The quotient of one quantity by another is the quotient of the first by the number of times that the second contains their common unit.

POSTULATE III.

Two quantities are to each other as their values.

THEOREM IX.

The sum of the equimultiples of several quantities is equal to the same multiple of their sum.

Let A, B and C be any quantities, and let A', B' and C' be equal to the respective products of A, B and C by *five*.

Because A' is equal to $A+A+A+A+A$ [*Gen. Def.* 25], B' to $B+B+B+B+B$, and C' to $C+C+C+C+C$, then A', B' and C' are together equal to $A+A+A+A+A+B+B+B+B+B+C+C+C+C+C$, or to $A+B+C+A+B+C+A+B+C+A+B+C+A+B+C$, that is, *five* times the sum of A, B and C; therefore the sum of A', B' and C' is equal to the product of the sum of A, B and C by *five*. But the same reasoning could be applied to any number of quantities and their respective products by any number; therefore the sum of the equimultiples of any number of quantities is equal to the same multiple of their sum, W. W. T. B. D.

COROLLARY. *The difference of the equimultiples of two quantities is equal to the same multiple of their difference.*

THEOREM X.

If two quantities have a common measure, they are to each other as the numbers of times they contain the common measure.

Let there be two quantities A and B, having a common measure M; let M be contained *seven* times in A and *three* times in B.

Let M be taken as the common unit of A and B; then the numbers *seven* and *three* are the values of A and B respectively [*Gen. Def.* 15]: therefore A is to B as *seven* is to *three* [*Post.* III], which W. T. B. D.

THEOREM XI.

(*Eucl.* V. 15.)

As two quantities are to each other, so are their equimultiples.

Let there be two quantities A and B, and let A' and B' be the respective products of A and B by *three*.

Let u be the common unit of A and B, and let it be contained *five* times in A and *seven* times in B; then A is to B as *three* is to *seven* [*Post.* III]. Because A is equal to $u+u+u+u+u$, the quantity A' is equal to *three* times $u+u+u+u+u$: but *three* times $u+u+u+u+u$ is equal to *three* times $u+$*three* times $u+$*three* times $u+$*three* times $u+$ *three* times u [ix]; therefore A' is equal to *five* times *three* times u. Likewise B' is equal to *seven* times *three* times u. Let *three* times u be taken as the common unit of A' and B'; then *five* and *seven* are their respective values, and A' is to B' as *five* is to *seven*: therefore A' is to B' as A is to B [i]. But the same reasoning could be applied if any other numbers were used instead of *five*, *seven* and *three*; therefore, as the quantities A and B are to each other, so are their equimultiples A' and B', which W. T. B. D.

THEOREM XII.

If two quantities contain as many equal parts as two others respectively, the four quantities are proportional.

Let two quantities A and B contain respectively *seven* and *three* parts equal to M, and let two other quantities C and D contain respectively *seven* and *three* parts equal to N.

Because M is a common measure to A and B, the quantity A is to C as *seven* is to *three* [x]; likewise C is to D as *seven* is to *three*: therefore A is to B as C is to D [i], which W. T. B. D.

COROLLARY. *If two quantities of one kind contain respectively as many equal parts as two other quantities of another kind, the four quantities are proportional.*

THEOREM XIII.

(*Eucl.* V. 16.)

If the terms of a proportion are quantities of the same kind, the homologous terms are proportional.

Let A be to B as C is to D.

Let m be the ratio of A and B; then m is also the ratio of C and D.

If m be equal to *one*, A is equal to B and C is equal to D; thence A is to C as B is to C: therefore A is to C as B is to D.

If m be greater than *one*, A is greater than B, and C is greater than D. Let u be the common unit of B and D, and let u be contained *three* times in B and *seven* times in D; then B is to D as *three* is to *seven* [*Post.* III]. Because B is equal to *three* times u or $u+u+u$, the quantity A is equal to m times $u+u+u$, that is, to m times $u+m$ times $u+m$ times u [ix]: therefore A is equal to *three* times m times u. Likewise C is equal to *seven* times m times u. Let m times u be taken as the common unit of A and C; then A is to C as *three* is to *seven*: therefore A is to C as B is to D [i]. But the same reasoning could be applied if any other numbers were used instead of *three* and *seven*: therefore the homologous terms are proportional.

If m be smaller than *one*, the same reasoning would prove that B is to D as A is to C; therefore, whatever be the ratio of A and B, the homologous terms of the proportion *A is to B as C is to D* are proportional, W. W. T. B. D.

THEOREM XIV.

The sums of the homologous terms of a proportion are to each other as two corresponding terms.

Let A be to B as C is to D.

Let m be the ratio of A and B; then m is also the ratio of C and D.

If m be equal to *one*, A is equal to B and C is equal to D; now, $A+C$ is evidently to $A+C$ as A is to A: therefore $A+C$ is to $B+D$ as A is to B.

If m be greater than *one*, A is greater than B and C is greater than D. Because A is equal to m times B, and C is equal to m times D, the quantities A and C are together equal to m times B together with m times D: but m times $B+m$ times D are equal to m times $B+D$ [ix]; then $A+C$ is equal to m times $B+D$; therefore $A+C$ is to $B+D$ as A is to B or as C is to D.

The same reasoning could be applied, if m were smaller than *one*; therefore the sums $A+C$ and $B+D$ are to each other as two corresponding terms, W. W. T. B. D.

Corollary I. *The sums of the homologous terms of a proportional series, are to each other as two corresponding terms* (Eucl. V. 12).

Corollary II. *The differences of the homologous terms of a proportion, are to each other as two corresponding terms.*

Theorem XV.

(*Eucl.* V. 18.)

The sums of the corresponding terms of a proportion are to each other as two homologous terms.

Let A be to B as C is to D.

Because A is to C as B is to D [xiii], the sum of A and B is to that of C and D as A is to C or as B is to D [xiv], which W. T. B. D.

Corollary I. *The differences of the corresponding terms of a proportion, are to each other as two homologous terms* (Eucl. V. 17).

Theorem XVI.

If two homologous terms of a proportion are equal to each other, the other terms are equal to each other.

Let A be to B as C is to D, and let B be equal to D.

Because A is to C as B is to D [xiii], the ratio of A and C is equal to that of B and D: but the ratio of B and D is equal to *one;* then that of A and C is also equal to *one;* therefore A is equal to C, which W. T. B. D.

Corollary. *If the consequents of a proportional series are equal to each other, the antecedents are equal to each other,* and conversely.

Theorem XVII.

If two proportions have three terms of the one equal to three homologous terms of another, each to each, the fourth term of the one is equal to that of the other.

Let A be to B as C is to D, and let A be to B as C is to E.

Because the ratios of C and D and of C and E are each equal to that of A and B, they are equal to each other [i]; thence C is to D as C is to E: therefore D is equal to E [xvi], which W. T. B. D.

THEOREM XVIII.

The product of a quantity by another is to the former as the latter is to their common unit.

Let *P* be the product of *A* by *B*, and let *B* be equal to *m* units.

Because *P* is equal to *m* times *A* [*Gen. Def.* 37], *P* is evidently to *B* as *m* times *A* is to *B*; thence *P* is to *B* as *m* times *A* is to *m* units. Because *A* is to the unit as *m* times *A* is to *m* units [xi], *P* is to *B* as *A* is to the unit [i]; therefore *P* is to *A* as *B* is to the unit [xiii], W. W. T. B. D.

COROLLARY. Conversely, *if a quantity is to another as a third is to their common unit, the first is the product of the second by the third.*

THEOREM XIX.

The product of a quantity by another is equal to that of the second by the first.

Let *P* be the product of *A* by *B*, and let *Q* be the product of *B* by *A*.

Because *P* is the product of *A* by *B*, *P* is to *A* as *B* is to *one* unit [xviii]; likewise *Q* is to *B* as *A* is to *one* unit, thence *Q* is to *A* as *B* is to *one* unit [xiii]: therefore *P* is equal to *Q* [xvii], which W. T. B. D.

COROLLARY. *The product of three quantities is the same whatever be the order in which they are multiplied.*

THEOREM XX.

The product of the extremes of a proportion is equal to that of the means.

Let *A* be to *B* as *C* is to *D*; let *P* be the product of the extremes, and let *Q* be that of the means.

Let *m* be the ratio of *A* and *B*; then *m* is also the ratio of *C* and *D*. Because *A* is equal to *m* times *B* and *C* is equal to *m* times *D*, the product of *A* by *D* is equal to that of *m* times *B* by *D*, and the product of *B* by *C* is equal to that of *B* by *m* times *D*: but the product of *m* times *B* by *D* is equal to that of *B* by *m* times *D* [xix]; therefore the product of *A* by *D* is equal to that of *B* by *C* [i], which W. T. B. D.

THEOREM XXI.

If the product of two quantities be equal to that of two thers, they are reciprocally proportional.

Let P be the product of A and D; let Q be that of B and C, and et P be equal to Q.

Let D be equal to m units; then P is equal to m times A [*Gen. Def.* 37]. Because Q is the product of C by B [xix], the product Q is to C as B is to *one* unit [xviii]; therefore P is also to C as B is to *one* unit: now B is to the unit as m times B is to m units, or D [xi]; then P is to C as m times B is to D [i]; therefore P is to m times B as C is to D [xiii]. Because A is to B as m times A is to m times B [xi], and m times A are equal to P, the quantity A is to B as P is to m times B; therefore A is to B as C is to D [i], which W. T. B. D.

THEOREM XXII.

If two continual proportions have the extremes of the one equal to those of the other, each to each, the means of the one are equal to those of the other.

Let A be to B as B is to D, and let A be to C as C is to D.

Because the product of B by B and that of C by C are each equal to that of A by D [xx], the product of B by B is equal to that of C by C [i]; thence B is to C as C is to B [xxi]: but B cannot be to C as C is to B unless the ratio of the corresponding terms be equal to *one*; therefore B is equal to C, which W. T. B. D.

PLANE GEOMETRY

IN FOUR BOOKS.

BOOK I.

ON LINES.

> Since the classifications in any Science are continually modified as scientific knowledge advances, the definitions in the sciences are also varying.”
>
> MILL.

PLANE GEOMETRY.

BOOK I.

ON LINES.

PRELIMINARY DEFINITIONS.

I. *LINES.*

> "A straight line is that of which the extremity hides all the rest, the eye being placed in the continuation of the line." PLATO.
>
> "A straight line is that which lies similarly between its extreme points." EUCLID.
>
> "A straight line is the shortest distance between two points." ARCHIMEDES.

ıere be imagined three points, the third of which is placed in the same
ı relatively to both the first and the second as the second is placed
ly to the first, the distance being disregarded, so that the figure
by the three points be similar [*Gen. Def.* 19] to that formed by the
o ; then let there be imagined a fourth point placed relatively to each
ɔreceding ones in the same position as the third is relatively to each
ə preceding it, and so on indefinitely; the idea of a line will thus be
ed, all points of which will proceed in the same way, and which will
ı *magnitude only* from the figure formed by the two first points, or
ı of the points : this line is called a *straight* line. Thence

A *straight* line is that which is similar to each of its
l

evident that a straight line may have any portion thereof expanded,
ɔed, without ceasing to be straight, for then it remains similar to

---

der the better to understand this definition, and all others of the same kind,
nust be imagined as great as it can be ; that is, infinite.

each of its parts; also, that the indefinite expansion of that portion reproduces the whole line.

Let there now be imagined two points, then a third one not in a straight line with the first two, and a fourth one placed in the same position relatively to the two last ones as the third is placed relatively to the first two; then a fifth point placed in the same position relatively to the last two, as the fourth is placed relatively to the two points preceding it, and so on indefinitely; if the distance between these points be imagined to be infinitely small, the idea of a line may thus be conceived which will differ, not only in magnitude, but also in form, from the figure formed by the first two points, or any two of the points: this line is called a *curved* line. A still more complex line would be obtained in considering more than three points at first, and in varying the relative position of the succeeding ones. Whence

2. A *curved* line, or simply *curve,* is that which is not similar to each of its parts.

It is obvious that there can be but two kinds of lines,—the straight line and the curve; and that there can be but one kind of straight line, whilst the forms of curved lines may be varied to infinity.

It follows from what precedes, that *two points* suffice to produce in the mind the idea of a straight line, and that no less than *three points* are required to produce that of a curve. It follows equally from the same, that a curved line consists of infinitely small straight lines.

II. *SURFACES.*

> "A plane surface is one whose extremities hide all the intermediate parts, the eye being placed in its continuation." PLATO.
>
> "A plane surface is that which lies similarly with the straight line in it." EUCLID.
>
> "A plane surface is that in which, any two points being taken, the straight line joining them lies wholly in that surface." HERON.

Let there be imagined a straight line moving indefinitely out of its length, so as to follow another straight line, and keeping constantly the same position relatively to it; the idea of a surface will thus be conceived, which will be similar to each of its parts, and is called a *plane* surface. Whence

3. A *plane* surface, or *plane,* is that which is similar to each of its parts.

Hence, the indefinite expansion of any portion of a plane reproduces the whole plane.

Let there now be imagined, a straight line, which, in its motion out of its length, follows a curve, it will then be possible to conceive the idea of a surface which is no longer similar to each of its parts; the same would happen if a curve were conceived to move on a straight line or another curve: this surface is called a *curved* surface. Whence

4. A *curved* surface is that which is not similar to each of its parts.

It is obvious that there can be but two kinds of surfaces, the plane and curved surfaces; and also that there can be but one kind of plane, whilst curved surfaces may be endlessly varied.

It follows from the preceding definitions that the idea of a plane is produced in the mind by two straight lines, whilst that of a curved surface is produced by curved lines only, or straight and curved lines together. It follows also from the same and from the definitions of the curved line, that a curved surface consists of infinitely small planes.

A plane is of its nature infinite, as is the straight line; it may contain points, straight lines, curved lines, and even finite portions of a plane; a curved surface may contain points, curved lines, finite portions of a curved surface, and sometimes straight lines, but no finite portion of a plane.

As a surface divides space into portions, these portions must be on different sides of the surface: the definition of a plane surface shows that it has only two sides.

5. The two sides of a plane are called its *faces*.

In the same manner a straight line contained in a surface has only two sides in that surface, and a portion of that surface on each side. Again, a point contained in a line has only two sides in that line and a portion of the line on each side.

6. A line, of which all points, and none else, have a common property, is called a *locus*.

III. *DIVISIONS OF GEOMETRY.*

Elementary Geometry is divided into two parts: *Plane* Geometry and *Solid* Geometry.

Plane Geometry has for its objects, lines contained in one and the same plane, and the portions of the plane limited by such lines.

The object *of Solid Geometry* are surfaces and geometrical bodies.

CHAPTER I.

ON STRAIGHT LINES.

SECTION I.

ON STRAIGHT LINES GENERALLY.

> "Mathematical proofs, like diamonds, are hard as well as clear, and will be touched by nothing but strict reasoning." LOCKE.

DEFINITIONS.

7. When the length of a straight line is known, the line is said to be given *in magnitude.*

A straight line is of its nature infinite in magnitude and cannot be measured; but a portion of a straight line may be measured by another portion. It is evident that the unit for measuring a straight line must be a straight line.[1]

8. If two lines have one point in common, they are said to *meet,* or *touch* each other; if, when meeting, one of the two lines has some of its points on both sides of the other, the first is said to *cut,* or *cross,* the second, and their common point, or intersection, is called their point *of section.*

It follows from the definitions of the plane and of the straight line, that no line contained in a plane can have any of its points on both sides of a straight line without first cutting it.

9. A *transversal* to two or more straight lines is a straight line which cuts them in distinct points.

10. The *meeting* point of two lines which do not touch

---

[1] A *portion* of an infinite straight line is generally called a *straight* line for the sake of brevity, but the mind must supply the rest of the expression when the sense of a sentence indicates that a portion only is meant.

each other, is the point in which they would meet if they were indefinitely produced.

11. The portion of a line intercepted between two portions of one or two other figures, is called an *intercept*.

12. The *middle* point of a line is that which divides it into two equal parts: the *middle* line of any two limited straight lines is that which joins[1] their middle points.

POSTULATES.

IV. *The elements of a straight line are all equal to another.*

V. *There is one, and only one, line which is the shortest that can join two points.*

VI. *Through any point there may be any number of straight lines of any magnitude.*

VII. *If a plane be folded upon itself, the two parts coincide entirely with each other.*

THEOREM 1.

All parts of a straight line are similar to each other.

For, if any two parts of a line had not the same form, the whole line could not have the same form as each of them, and therefore would not be a straight line [*Def.* 1].

COROLLARY. *The direction of a straight line is the same throughout its extension.*[2]

---

[1] Whenever in The Elements the *joining* of two points is spoken of, the meaning is that one point is to be joined to the other by a *straight line.*

[2] Let the student demonstrate every corollary, and find out the converses and inverses of every *proposition; also demonstrate* that they are true or false as the case may be.

THEOREM 2.

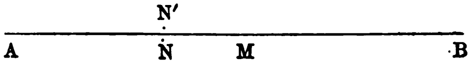
A straight line turning upon two of its points, has none of its parts out of its first place.

For each part of a straight line being similar to the whole line [*Def.* 1], the figure formed by each part when turning must necessarily be the same as that formed by the whole line. Inversely

If a line, when turning upon two of its points, has any of its parts out of its first place, the line is not straight.

THEOREM 3.

From any point to another there cannot be more than one straight line.



Let A M B and A N B be two straight lines having their extremities at A and B.[1]

Let the plane containing the two lines be folded upon the straight line A M B as an axis; then no part of A M B will be out of its first place [2]. Should any point N of the other line be out of A M B, that is, on one side thereof, then it would, after the folding, have a place N′ on the other side, different from N, and A N B would not be a straight line [2]; but A N B is a straight line [*Hyp.*]; therefore no point of it is out of A M B; that is, the two lines A M B and A N B are one and the same straight line, which was to be demonstrated. Inversely

If one of two distinct lines joining two points be straight, the other line is not straight.

COROLLARY I. *Two straight lines having the same extremities are not distinct from each other.*

COROLLARY II. *If a plane contain the two extremities of a straight line, it contains the whole line.*

---

[1] It is understood that the two lines are contained in the same plane, as all figures throughout Plane Geometry.

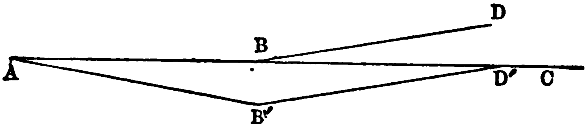
Corollary III. *If the extremities of two equal straight lines coincide, their middle points coincide.*

Scholium. The magnitude of a straight line is determined when its extremities are known.

Theorem 4.

Two straight lines which have two points in common, coincide with each other throughout their whole extension.

Let A B C and A B D be two straight lines coinciding in A and B.



The parts A B of the two lines coincide with each other, for they have the same extremities A and B [3 i]. The parts B D and B C coincide also, for if the line A B D, be turned on the point A until the point D coincide with a point D′ of A B C, there would be two straight lines A B D′ and A B′ D′, having the same extremities A and D′ and not coinciding, which is impossible [3]; therefore the two straight lines, A B C and A B D, coincide with each other throughout their extension, W. W. T. B. D. Inversely

Two distinct straight lines cannot meet in more than one point.

Corollary I. *If two straight lines coincide with two others, the meeting point of the two first coincides with that of the others.*

Corollary II. *If any number of straight lines*[1] *meet in one point, each one has no other point in common with the others.*

Corollary III. *If a plane indefinitely extended contain two points of a straight line, it contains the whole line.*

Corollary IV. *All straight lines are similar to one another.*

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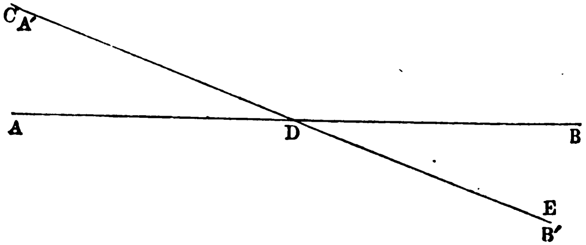
[1] When two, or any number of figures are spoken of in The Elements, distinct figures are always *meant unless the contrary* be stated.

THEOREM 5.

If two straight lines meet, they cut each other.

Let A B be a straight line, and C D E another straight line, meeting A B in a point D.



Let the straight line A B turn upon the point D, until the point A has reached some point A′ of C D. When this motion begins, the portion D B of A B is removed farther from D C, until it has reached a position D B′ on the side of A B opposite to C D. Because the two straight lines A′ D B′ and C D E have the two points A′ and D in common, they coincide with each other throughout their extension [4]: that is, D B′ is the real position of the continuation of C D; therefore C D E has the position C D B′, and cuts A B in D, which W. T. B. D. Inversely

If two straight lines, contained in the same plane, cannot cut each other, they cannot meet.

THEOREM 6.

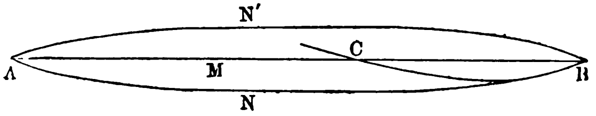
The straight line is the shortest that can join its extremities.

Let A M B be a straight line and let A N B be the shortest line that can join the points A and B.

Should A N B not coincide with the straight line A M B, it would be all or partly on one side thereof.

Let it be supposed, first, that A N B is all on one side of A M B. Let the plane containing the two lines be folded upon the straight line A M B as an axis. No part of A M B will move out of its first

ition [2], and A N B will take a position A N′ B on the other side
A M B; thence the line A N′ B is equal to the line A N B: that is,
N B is not less than A N′ B; therefore, if A N B were all on one
e of the straight line A M B, it would not be the shortest line
t can join A and B [*Post.* V.]



et it now be supposed that A N B is partly on one side of A M B
partly on the other; then A N B cuts the straight line in one
t at least. Let C be the point of section; then A N B would not
he shortest line that can join B and C [*Dem.*], still less the shortest
can join B and A: therefore, if A N B were partly on one side
A M B and partly on the other, it would not be the shortest line
can join A and B.

he same reasoning would prove that if A N B were partly to
cide, and partly not to coincide, with the straight line A M B, it
ld not be the shortest line that can join A and B.

en, if A N B did not coincide throughout with the straight line
B, it would not be the shortest line that can join A and B; but
B is the shortest line that can join A and B [*Hyp.*]: therefore
B coincides throughout with A M B; that is, the straight line
B is the shortest that can join its extremities. It is evident that
same construction and reasoning would prove that any other
ght line than A M B is the shortest that can join its extremities,
W. T. B. D. Inversely,

*y line which is not the shortest that can join its extremities, is not
ght.*

ROLLARY I. *If three points be joined by straight lines, each to the
two, one line shall be less than the two others together.*

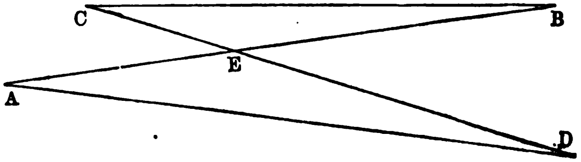
ROLLARY II. *If the lines joining a point to the extremities of a
ght line are together equal to the latter, the point is on the straight*

Scholium. The distance between two points is the length of the straight line joining them (*Gen. Def.* 8).

THEOREM 7.

Two straight lines which cut each other are together greater than the lines joining their opposite extremities.

Let A B and C D be two straight lines cutting each other in E, and let A D and B C join their opposite extremities.



The lines A E and E D are together greater than A D, and the lines E B and E C are together greater than B C [6 i]; therefore A E, E D, E B and E C are together greater than A D and B C [iii]. But A E and EB together form the line A B, and E D and E C together form the line C D; therefore A B and C D are together greater than A D and B C, which W. T. B. D.

COROLLARY I. *Each of two equal straight lines which cut each other is greater than each of equal lines joining their opposite extremities.*

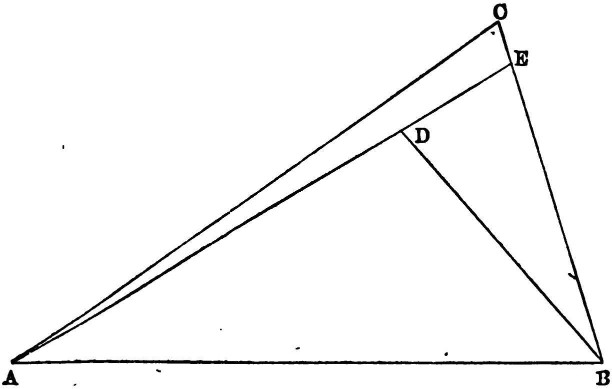
COROLLARY II. *If each of two points be joined to both extremities of a straight line by lines each equal to the latter, the line joining them is greater than each of the others.*

THEOREM 8.

(*Eucl.* I. 21.)

If three points be joined by straight lines, each to the others, two straight lines joining a point within the figure thus formed to the extremities of one of the first lines, are together less than the two other lines.

Let the points A, B and C be joined by the straight lines A B, A C d B C, and let a point D within the figure thus formed be joined the extremities of the line A B.



Let A D be produced to E. Then the straight line A E, or A D d D E, is less than A C and C E together [6 i]; also D B is less an D E and E B together: therefore, A D, D E and D B are together ss than A C, C E, D E and E B [v]. But C E and E B together rm the line C B; then A D, D E and D B are together less than C, C B and D E; therefore, A D and D B are together less than C and C B [iii], which W. T. B. D.

COROLLARY. *If three points be joined by straight lines, each to the hers, and if three points within the figure thus formed be joined by raight lines in the same way, the inner lines are together less than e outer lines.*

SECTION II.

ON PERPENDICULARS.

DEFINITIONS.

13. A straight line is said to be given *in position,* when s position is known relatively to one of its points.

It follows from the definition of the straight line that its position in relaon to one *of its points is the same* relatively to every other of them [1 i].

whence, the direction of a straight line is determined when its position is known relatively to one of its points; that is, when it is given in position.

It is evident that two straight lines cutting each other have not the same direction, since they have not the same position relatively to the point of section; thus, the direction of a straight line distinguishes it from any other line cutting it. It is also evident that if one straight line cutting another, changes its position relatively to the point of section, the first only ceases to cut the second when the two lines coincide with each other: whence, if two straight lines, differing however slightly in direction, have a point in common, they cut each other, for a second point in common would make them coincide [4], and give them the same direction.

If two straight lines cut each other, and one of them be conceived to turn on the common point of section so as to coincide with the other; the two lines have then the same position relatively to that point and, consequently, the same direction. Now, let the same line be conceived to turn back towards its first position, and then to continue that motion (constantly changing its position relatively to the point of section) until it again coincides with the other line, having then a position relatively to the same point, opposite to that of the stationary line. The two lines now have opposite positions relatively to their former point of section; they have consequently opposite directions. But between the two last positions, opposite to each other, there is a third position as different from one as from the other, and called the *contrary* position. Whence,

14. Two straight lines are said to have *contrary* directions when they are as far from having the same direction as they are from having opposite directions.

Hence, straight lines having contrary directions cannot have more than one point in common, whilst straight lines having opposite directions may coincide entirely, their opposite directions indicating only the manner in which they have been made to coincide. Thus, when two limited straight lines having a common extremity, have opposite directions, the continuation of each beyond that extremity coincides with the other line, and has the same direction; that is, two infinite straight lines which coincide, have the same direction, although portions of them may have opposite directions.

If a limited straight line meeting an indefinite straight line, be turned upon the point in common so as to be brought near coinciding with one of the two portions of the indefinite line, it is said to *incline* towards that portion, and to incline the *more* as the two directions differ the less.

15. Two straight lines are said to be *perpendicular to* or *n one another,* when they have contrary directions; each of them is called a *perpendicular.*

That is when no portion of one is inclining towards any portion of the other.

16. Two straight lines are said to be *oblique to* or *on one another,* when they have not the same direction, and neither opposite nor contrary directions; each of them is called an *oblique.*

That is when a portion of the one is inclining towards a portion of the other. It is obvious that two straight lines cutting each other are either perpendicular or oblique to one another.

17. The intersection of two straight lines is called the *foot* of each portion of the perpendicular or of the oblique.

If a perpendicular, or an oblique, to a straight line, do not touch it, its foot is determined by the intersection of its continuation with that of the straight line.

Since there may be, on any point of a straight line, another straight line cutting it, it is evident that there may be a perpendicular on any point of a straight line.

18. The perpendicular on the middle point of a straight line, is called the *middle* perpendicular on that line, or its *middle* perpendicular.

19. The *projection* of a point on a straight line is the foot of the perpendicular drawn from the point to the line. The *projection* of a straight line on another is the distance between the projections of its extremities on the latter.

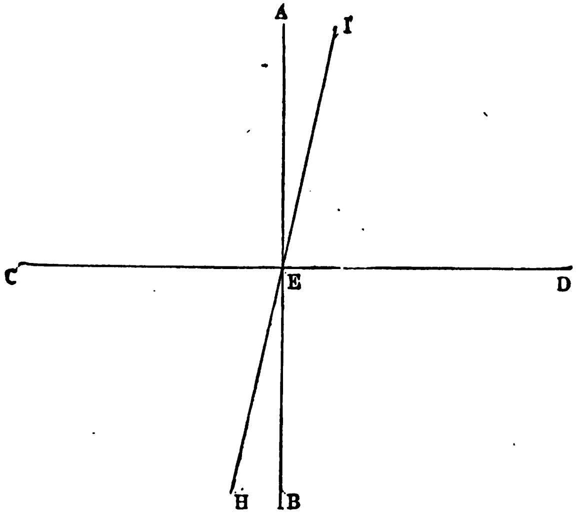
POSTULATE VIII.

From any point without a straight line there may be a perpendicular to it.

THEOREM 9.

Through a point of a straight line, there cannot be more than one perpendicular to it.

Let A B be a straight line perpendicular on C D through the point E.



The position of A B relatively to the point E is contrary to that of C D relatively to the same point [*Def.* 15]: that is, the direction of A E differs equally from those of E C and E D [*Def.* 14]. Any other straight line I H cutting C D at the point E has on one side of A E a portion I E, the direction of which differs more from that of E C than from that of E D; therefore I H will be an oblique to C D [*Def.* 16]. As the same reasoning could be applied to any straight line cutting C D through E, but A B itself, it is evident that through the point E there can be but one perpendicular to C D. The same construction and reasoning could be applied to a perpendicular through any other point than E of C D, or of any other straight line; therefore, through any point of any straight line, there can be but one perpendicular to it, W. W. T. B. D. Inversely

e of several straight lines cutting another at the same point is a
dicular to the latter, all the others are oblique to the same.

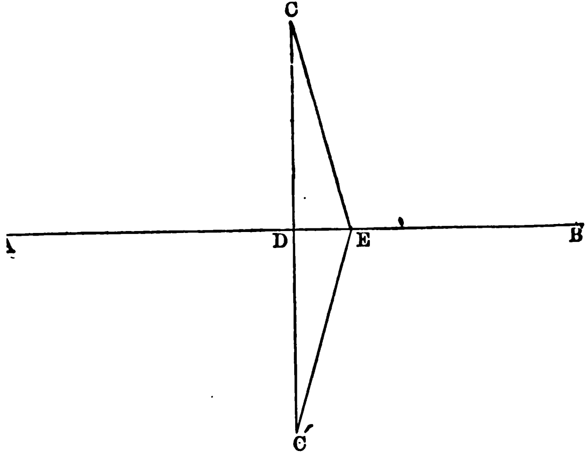
LLARY I. *If a plane be folded upon a straight line, every*
dicular to the latter coincides with its continuation beyond its
d conversely.

LLARY II. *If through a point of a straight line, a perpendicular*
ed to it on each side, each one is the continuation of the other.

THEOREM 10.

m a point without a straight line, no more than one per-
ular can be drawn to it.

D be a perpendicular to the straight line A B from the point C.
E be another perpendicular to A B drawn from the same



without A B, and let C D be produced until D C′ is equal to
ow let the plane containing the figure be folded upon A B as
s: then D C will coincide with its continuation D C′ [9 i]
e point C with C′, and C E will have the position C′ E. But
a perpendicular on A B [*Const.*]; then C′ E is also perpendicu-
A B, and is the continuation of C E [9 ii]: therefore C E C′ is

a straight line. But there is already C D C′ a straight line having the same extremities; therefore C E C′ coincides with C D C′ [3]: that is, C E is none other than the perpendicular C D itself and is not distinct from it,[1] W. W. T. B. D. Inversely

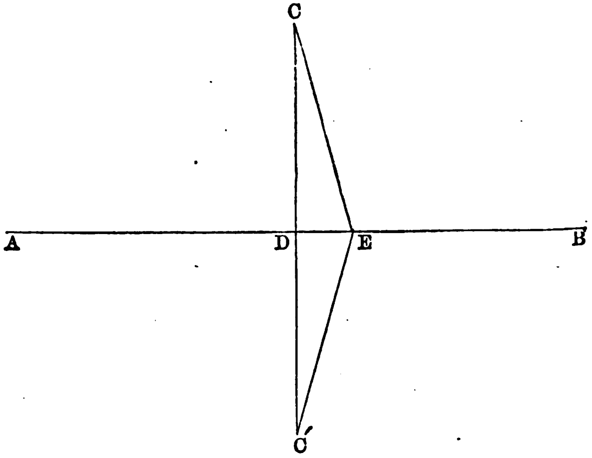
If one of several straight lines joining a point to an indefinite straight line, be perpendicular to the latter, the others are oblique to the same.

COROLLARY. *All perpendiculars to a straight line from every point of an oblique to it, are on the same side of the oblique.*

Scholium. A perpendicular to a straight line is determined when one of its points is known.

THEOREM 11.

Of all straight lines drawn to an indefinite straight line from a point without it, the perpendicular is the shortest.



Let C D be a perpendicular drawn to the straight line A B from a point C without it.

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[1] Let the student supply here, as in most other demonstrations, the rest of the reasoning suppressed for the sake of brevity.

Let C E be an oblique to A B from the same point C, and let C D be produced to C′ until D C′ is equal to D C, and let the points C′ and E be joined; now let the plane containing the figure be folded upon A B as an axis: then D C will coincide with D C′ [9 i] and C with C′. Because the lines C E and C′ E have the same extremities, C E and C′ E are equal to each other [3 i]; but C D C′ is a straight line, then C D C′ is less than C E and C′ E together [6 i], therefore half of C D C′, or C D, is less than C E [vii]. The same construction and reasoning would prove that C D is shorter than any other oblique drawn from the point C, therefore C D is the shortest line that can be drawn from the point C to the line A B, which W. T. B. D. Inversely

If a straight line is not the shortest joining a point to an indefinite straight line, it is not perpendicular on the latter.

COROLLARY I. Conversely, *if a straight line is the shortest that can join a point to an indefinite straight line, it is perpendicular on the latter.*

COROLLARY II. *The greater of two unequal straight lines joining a point to an indefinite straight line, is an oblique to the latter.*

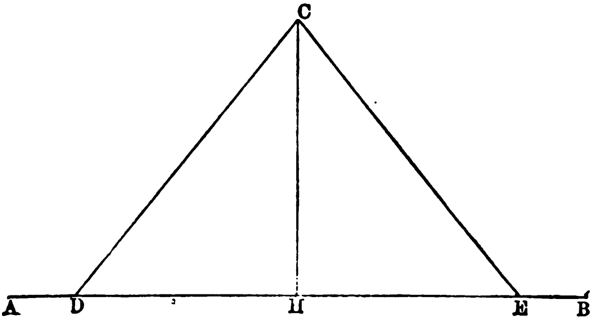
Scholium. The distance from a point to a straight line is the length of the perpendicular drawn from the point to the line (*Gen. Def.* 8).

THEOREM 12.

Two obliques to a straight line from a point without it, which have equal projections on the latter, are equal to each ther.

Let C D and C E be two obliques on A B; let C H be a perpenicular on A B, and let D H be equal to H E.

Let the plane containing the figure be folded upon C H; hen H B, perpendicular on C H, will coincide with its coninuation [9 i]. Because H E is equal to H D [*Hyp.*], the point E will coincide with D; thence the two obliques C E and C D have now



the same extremities: therefore they are equal to each other [3i], W. W. T. B. D. Inversely

Two unequal obliques to a straight line from a point without it have unequal projections.

THEOREM 13.

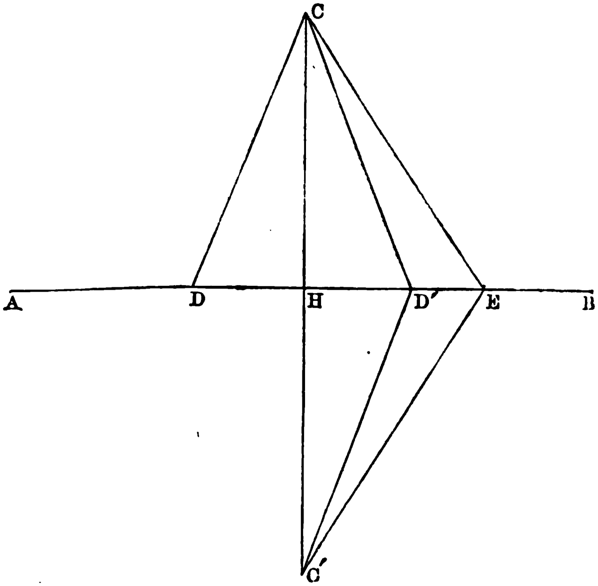
Two obliques to a straight line from a point without it, which have unequal projections on the latter, are unequal.

Let C D and C E be two obliques on the straight line A B; let C H be a perpendicular on A B, and let the projection H E of one be greater than the projection H D of the other.

Let H D′ be equal to H D; let also C be joined to D′; then C D will be equal to C D′ [12].

Let also C H be produced to C′ until H C′ is equal to H C, and let the point C′ be joined to E and D′. Now let the plane containing the figure be folded upon A B, then H C will coincide with H C′ [9 i] and the point C will coincide with C′. Because the lines C E and C′ E, have the same extremities, they are equal to each

r [3 i]; also C D′ is equal to C′ D′. Now C E and E C′ are to-
er greater than C D′ and D′ C′ [8]; therefore C E is greater than

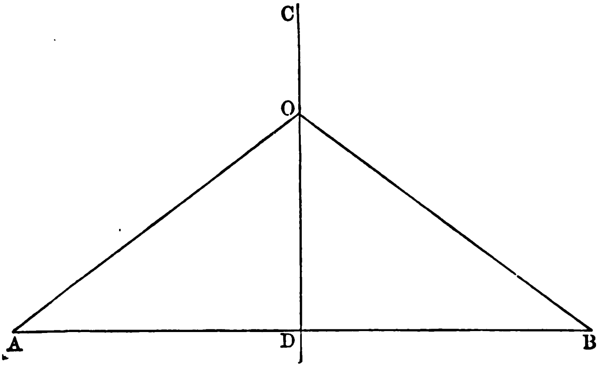


[vii]: but C D′ is equal to C D [*Dem.*]; therefore C E is greater
C D, which W. T. B. D. Inversely
o equal obliques to a straight line from a point without it, have
l projections on the latter. (Conv. of 11.)
ROLLARY I. *Of two obliques to a straight line from a point without*
e greater is that which has the greater projection, and conversely.
ROLLARY II. *A straight line cannot be joined to a point without it*
ore than two equal straight lines.
ROLLARY III. *If a point is equally distant from the extremities*
straight line, it is less distant from any other point of the same

THEOREM 14.

Each point of the middle perpendicular on a straight line is equally distant from the extremities of the latter.

Let C D be the middle perpendicular on the straight line A B.



Let any point O of C D be joined to A and B. The two obliques O A and O B, having equal projections D A and D B [*Hyp.*], are equal to each other [12], therefore the point O is equally distant from the extremities of A B [*Gen. Def.* 8]. As the same construction and reasoning could be applied to any other point in C D, it is evident that each point of C D. is equally distant from the extremities of A B, which W. T. B. D. Inversely

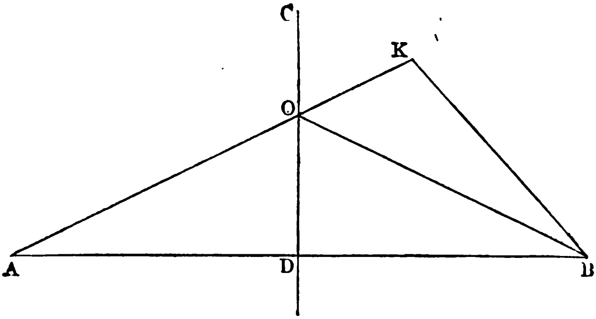
Any point which is not equally distant from the extremities of a straight line, does not belong to its middle perpendicular.

THEOREM 15.

Any point without the middle perpendicular on a straight line is not equally distant from the extremities of the latter.

Let C D be the middle perpendicular on a straight line A B.

Let any point K without the line C D be joined to A and B; then K A will cut C D [*Def.* 8] in a point O.



Let the points O and B be joined. The straight line K B is less than K O and O B together [6 i]; but O A is equal to O B [14]; therefore K B is also less than K O and O A, or K A: that is, the point K is not equally distant from the extremities of A B, which W. T. B. D. Inversely

Every point equally distant from the extremities of a straight line belongs to its middle perpendicular. (Conv. of 13.)

Corollary I. *A point placed between one extremity of a straight line and its middle perpendicular is nearer to that extremity than to the other.*

Corollary II. *A straight line having two of its points equally distant from the extremities of another, is the middle perpendicular on the latter.*

Corollary III. *A straight line is bisected by the perpendicular, one point of which is equally distant from its extremities.*

Scholium. The middle perpendicular on a straight line is the locus of all points equally distant from the extremities of the latter (*Def.* 6).

SECTION III.

ON PARALLELS.

DEFINITIONS.

> "Geometers have usually preferred to define parallel lines by the property of being in the same plane and never meeting. This, however, has rendered it necessary for them to assume, as an additional axiom, some other property of parallel lines; and the unsatisfactory manner in which properties for that purpose have been selected by Euclid and others has always been deemed the opprobrium of elementary geometry."
>
> MILL.

When two straight lines coincide with each other, they have the same direction, but are not distinct; they may however be conceived to be separated from one another, but kept in the same position relatively to each of their points, that is, with the same direction; they are then said to be *parallel to one another*. Whence

20. Straight lines *parallel to one another*, or *parallels*, are straight lines having the same direction.

It has been seen that two straight lines which coincide are said to have either the same direction or opposite directions, according to the manner they are supposed to have been made to coincide; so also two straight lines which do not coincide, but have the same direction, are said to have either the same direction or opposite directions, according to the manner they are supposed to have been brought into their respective positions; thus, two straight lines having opposite directions may also be said indifferently to have the same direction or opposite directions; consequently they are parallel.

It follows from this definition that two parallels cannot have one common point without coinciding throughout their extension; whence, straight lines cutting each other are not parallel, which is already evident from the fact that they have not the same position relatively to the point of section, and consequently, have not the same direction. It follows also that two distinct parallels remain throughout on the same side of each other.

It is obvious that when two straight lines contained in the same plane are neither perpendicular nor parallel to one another, they are oblique. *Perpendicularity*, *obliquity*, and *parallelism* are thus the only three positions that two straight lines may assume relatively to one another: of these, perpendicularity and parallelism are the most remarkable.

21. The *like* extremities of two parallels are those which

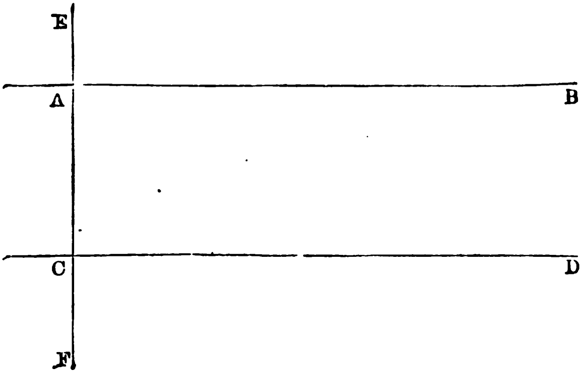
are on the same side of a transversal; the *unlike* extremities are those which are not on the same side of a transversal.

It is evident that straight lines joining the unlike extremities of two parallels cut each other between these parallels; the straight lines which join their like extremities, although their prolongations may cut each other, have no point in common between the parallels.

THEOREM 16.

Two perpendiculars on the same straight line are parallel to one another.

Let the straight lines A B and C D be both perpendicular to the indefinite straight line E F.



Let the intercept A C be reduced until the points A and C coincide, then the straight line E F will not have changed in position; but the perpendiculars A B and C D will coincide with each other [9]: therefore they have the same direction; that is, they are parallel to each other, W. W. T. B. D. Inversely

If one of two straight lines oblique to each other be perpendicular to a third, the other is not perpendicular to the same.

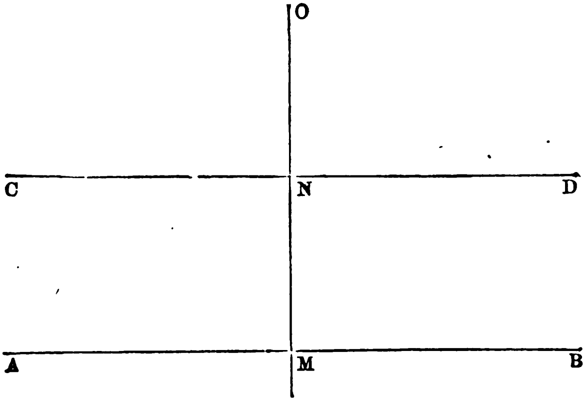
COROLLARY I. *A perpendicular and an oblique to the same straight line are oblique to each other.*

COROLLARY II. *From any point without a straight line a parallel may be drawn to the latter* [Post. VIII].

THEOREM 17.

A perpendicular to a straight line, is perpendicular to its parallel.

Let A B and C D be two straight lines parallel to each other, and let O M be a perpendicular to A B and cut C D in N.



If O M and C D were not perpendicular, then C D and A B would not be parallel to each other [16 i]: which is impossible [*Hyp.*]; therefore O M and C D are perpendicular to one another, W. W. T. B. D. Inversely

An oblique to a straight line is oblique to its parallel.

COROLLARY I. *Two parallels, however far produced, do not meet each other* (10).

COROLLARY II. *If one of two parallels be perpendicular to a straight line, the other is perpendicular to the same.*

COROLLARY III. *Two parallels to the same straight line are parallel to each other.* (Eucl. I. 30.)

COROLLARY IV. *No more than one parallel can be drawn to a straight line through a point without it* (9).

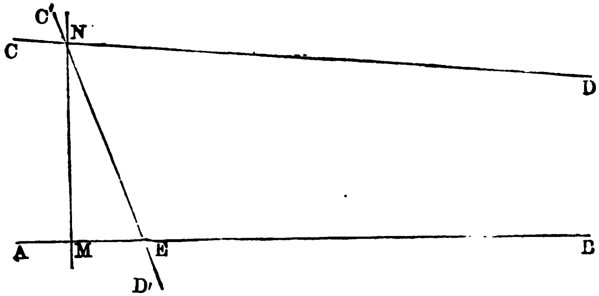
Scholium. A parallel to a straight line is determined when one of its points is known.

THEOREM 18.

(*Eucl. Ax.* 12.)

A perpendicular and an oblique to the same straight line, when indefinitely produced, cut each other.[1]

Let A B be a perpendicular to M N, and C D an oblique to the same.



Let C D be conceived to turn on the point N so that it inclines more and more on M N. When C D coincides with N M, it cuts A B in M. Let C D, still turning on the point N, move back to a position C′D′ between N M and its first position; it will then cut A B in a point E distinct from M.

Now let C D move back to its first position. As long as its direction is different from that of A B, that is as long as it is not perpendicular to M N, it will cut A B; for at whatever point C D should be conceived to leave A B, these two straight lines would have one point in common and differ in direction: therefore they would still cut each other [5]. But C D is not perpendicular to M N [*Hyp.*]; therefore, if indefinitely produced, it will cut A B, which W. T. B. D. Inversely

A straight line is perpendicular to another if, however far produced, it cannot meet another perpendicular to the same.

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[1] The student must here, as throughout Plane Geometry, supply the condition that these lines be contained *in the same plane.*

Corollary I. *Two straight lines not parallel to one another, when indefinitely produced, cut each other.*

Corollary II. *The perpendiculars to straight lines oblique to one another, when indefinitely produced, cut each other.*

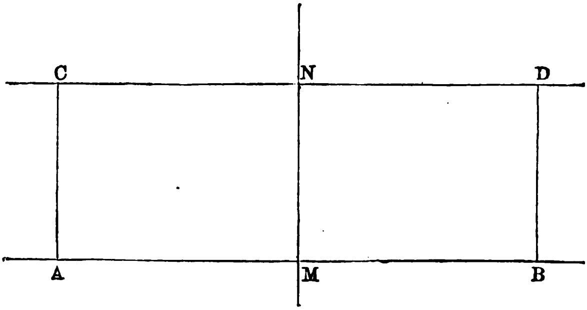
Corollary III. *If a straight line cut another, the first, when indefinitely produced, cuts every parallel to the second.*

Scholium. Two parallels meet at an infinite distance.

Theorem 19.

Two parallels are equally distant from each other throughout their extensions.

Let A B and C D be two parallels.



From any point C and D of C D let there be drawn to A B two perpendiculars C A and D B: the lengths of these two lines will be the distances from the points C and D to A B [11]. Let also M N be the middle perpendicular on the intercept A B, cutting C D at the point N [18 iii]: it will also be perpendicular on C D [16].

Now let the plane containing the figure be folded upon M N. Then M B will coincide with M A [9 i] and A with B, also N D with N C [9 i]. Because the straight lines M A and M B coincide with each other and become one and the same line, the perpendiculars D B and C A coincide with each other [9], and the point D, already coinciding with some point in N C, coincides with the point C.

ause the perpendiculars D B and C A have now the same ex- ities, they are equal to each other [3 i]; therefore the points C D are equally distant from A B. As the same construction and oning would prove that any other points of C D are equally ant from A B, and also that all points of A B are equally dis- from C D, it is evident that the parallels A B and C D are ughout equally distant from each other, W. W. T. B. D. In- ely

straight line having points unequally distant from another, is not llel to the latter.

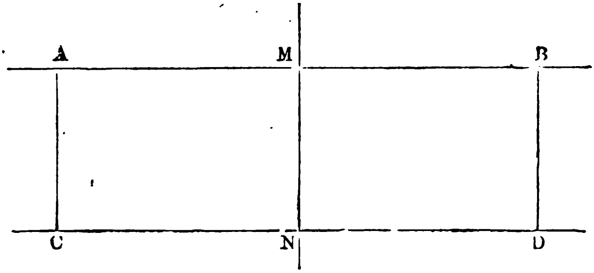
ROLLARY I. *The intercepts of two parallels between two per- iculars are equal to each other.*

ROLLARY II. *If two parallels be cut by two perpendiculars, the lle perpendicular on one of the intercepts bisects the opposite one.*

THEOREM 20.

straight line which has two of its points equally distant another, is parallel to the latter.

t A B be a straight line, having two of its points A and B lly distant from the straight line C D.



t A C and B D be perpendicular to C D; then they are equal to other [11 i]. Let M N be the middle perpendicular on the inter- C D, and let the *plane containing* the figure be folded upon M N then N D will coincide with N C [9 i], and, as they are equal to each other, the point D will coincide with the point C. Because the straight lines N D and N C coincide and form then one and the same line, the perpendicular B D will coincide with A C [9], and, as they are equal to each other, the point A will coincide with the point B. Because M A and its continuation M B have now the same extremities, they coincide; thence A B is perpendicular to M N [9 i]: therefore A B and C D, being both perpendicular to M N, are parallel to each other, W. W. T. B. D.

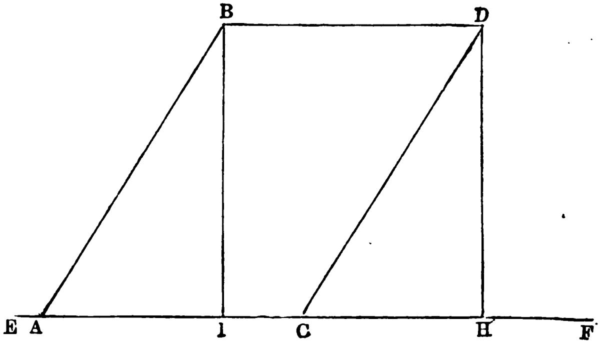
COROLLARY I. *If two parallels be cut by two perpendiculars, the middle line of two opposite intercepts is parallel to the others.*

COROLLARY II. *If two parallels be cut by two perpendiculars, the middle lines of opposite intercepts are perpendicular to each other, and bisect each other.*

THEOREM 21.

The straight lines joining the like extremities of two equal parallels, are parallel.

Let A B and C D be two equal parallels, and let their like extremities be joined by the straight lines B D and A C.

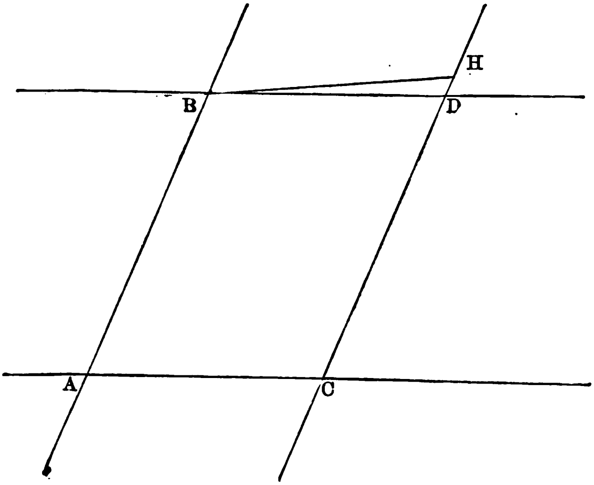


Let A C be produced on both sides of the parallels to E and F, and let there be drawn from B and D two perpendiculars B I and D H on E F. Let the intercept A C be reduced until the points C and A coincide with each other [*Gen. Def.* 22]; the parallels A B and C D, having then one point in common, will coincide with each other. The lines A B and C D being equal to one another [*Hyp.*], the points B and D will coincide with each other, and also the perpendiculars D H and B I on E F [10]; the lines D H and B I will then meet E F at the same point: that is, the points H and I will coincide with each other. Because the perpendiculars D H and B I have the same extremities, they are equal to each other [3 i]; thence the straight line B D has two of its points, B and D, equally distant from A H: therefore B D is parallel to A H [20], which W. T. B. D.

COROLLARY. *Two equal parallels have equal projections on the straight lines which pass through their like extremities.*

THEOREM 22.

Conversely, *the intercepts of two parallels between two parallels are equal to one another.*



Let A B and C D be the intercepts of two parallels between the parallels A C and B D.

Let C H be equal to A B, and let B H be joined; then B H is parallel to A C [21]. But B D is parallel to A C [*Hyp.*], therefore B H and B D coincide with each other [17 iv], and also D and H: that is C D is equal to C H. But C H is equal to A B [*Const.*], therefore C D and A B are equal to one another, W. W. T. B. D.

Corollary I. *The intercepts of any number of parallels between two parallels are equal to one another.*

Corollary II. *The lines joining the like extremities of two equal parallels, are equal to each other.*

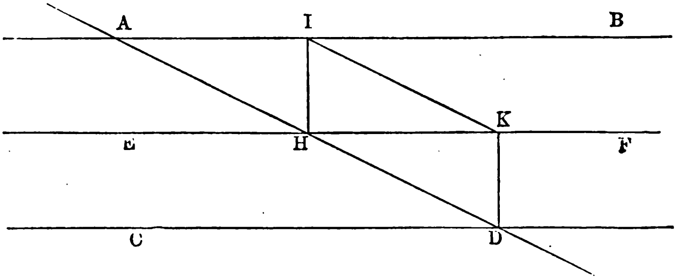
Corollary III. *The parallel to one of the intercepts of two parallels cut by two other parallels, through the middle point of one of the adjoining intercepts, bisects the other adjoining one.*

Corollary IV. *If two parallels be cut by two other parallels, the middle line of two opposite intercepts is parallel and equal to the others.*

Corollary V. *If two parallels be cut by two other parallels, the middle lines of opposite intercepts, bisect each other.*

Theorem 23.

If two parallels be equally distant from a third parallel, the intercepts of a transversal between the three parallels are equal to each other.



Let A B and C D be two parallels equally distant from E F, parallel to them, and let A H and H D be the intercepts of a transversal A D between the three parallels.

Let H I and D K be two perpendiculars to E F, through the points H and D, and let I K be joined. Because two parallels are throughout equally distant from each other [19], and E F is equally distant from A B and C D, the line H I is equal to D K; because H I and D K are both perpendicular to E F, they are parallel to each other [16]: therefore I K is parallel [21] and equal [22 ii] to H D. Because I K is parallel to A D, the intercepts I K and A H are equal to each other [22]; therefore H D is equal to A H [i], which W. T. B. D. Inversely

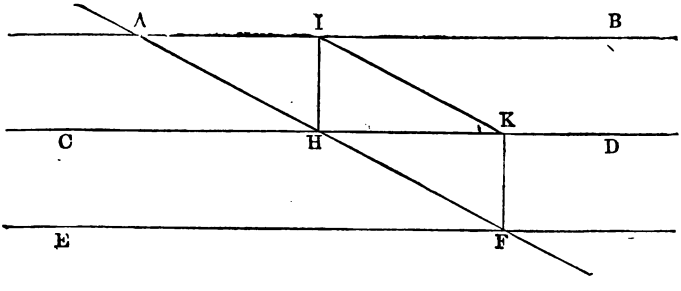
If the intercepts of a transversal between three parallels be unequal, the middle parallel is not equally distant from the others.

Corollary I. *The intercepts of a transversal between any number of equally distant parallels are equal to one another.*

Corollary II. *The point of the intercept of a transversal between two parallels, which is equally distant from both parallels, is the middle point of the intercept.*

Theorem 24.

Conversely, *if the intercepts of a transversal between three parallels be equal to each other, the middle parallel is equally distant from the others.*



Let A B, C D and E F be three parallels, and let the intercepts

A H and H F of a transversal A F between them be equal to each other.

Let F K be a perpendicular on C D; let K I be a parallel to A F through the point K, and let H I be joined. Because the intercepts A H and I K are equal to each other [22] and H F is equal to A H [*Hyp.*], the line I K is equal to H F [i]; therefore H I and F K are parallel [21] and equal [22 ii] to each other. But F K is perpendicular to C D [16]; then H I is also perpendicular to C D [17 ii]: therefore C D is equally distant from A B and E F [11], which W. T. B. D. Inversely

If two parallels be unequally distant from a third parallel, the intercepts of a transversal between the three parallels are unequal.

Corollary I. *If two parallels unequally distant from a third parallel be cut by a transversal, the intercept of the latter between the third and the more distant parallel is greater than that between the third and the less.*

Corollary II. *If the intercepts of a transversal between any number of parallels be equal to each other, the parallels are equally distant from each other.*

Corollary III. *The middle line of the intercepts of two tranversals between two parallels, is parallel to the latter, and equally distant from both of them.*

Corollary IV. *A straight line joining two parallels bisects every other straight line joining the parallels through its middle point.*

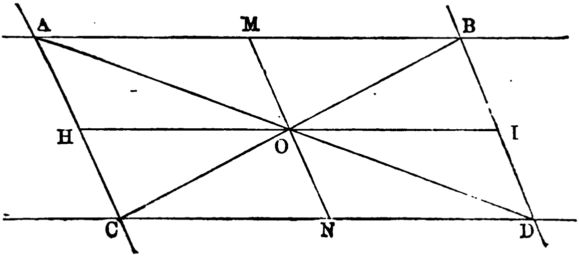
Corollary V. *Two parallels are equally distant from the middle point of the intercept of a transversal between them.*

Theorem 25.

If two parallels be cut by two other parallels, the lines joining the unlike extremities of two opposite intercepts bisect each other.

Let A B and C D be the intercepts of two parallels, between the parallels A C and B D.

Let A D and B C be joined; let H I be the middle line of the intercepts A C and B D, and M N that of the intercepts A B and C D;



and let O be the intersection of H I with A D: then H I bisects A D [24 iii]; that is, the point O is the middle point of A D. Because M N is the middle line of the intercepts A B and C D, the line M N bisects A D, that is, cuts A D at the point O; therefore the point O is the intersection of H I and M N. The same reasoning would prove that B C is also bisected by H I and M N at their point of section, that is in the same point O; therefore the lines A D and B C bisect each other, W. W. T. B. D. Inversely

If two straight lines cut each other in unequal parts, the lines joining their opposite extremities are not parallel to each other.

Corollary I. *The lines joining the unlike extremities of two equal parallels bisect each other.*

Corollary II. *The line joining unlike extremities of two equal parallels, bisects their middle line.*

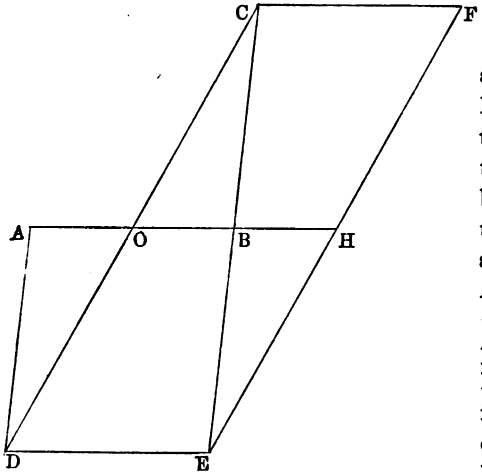
Corollary III. *The middle points of two equal parallels, and of the lines joining their unlike extremities, are points of the same straight line.*

Corollary IV. *If two parallels be cut by two perpendiculars, the lines joining the unlike extremities of their intercepts are equal to each other.*

Theorem 26.

If two straight lines bisect each other, the lines joining their opposite extremities are parallel to each other.

Let A B and C D be two straight lines bisecting each other in O, and let A D and C B join their opposite extremities.



Let D E be parallel and equal to A B, and let B E be joined; then B E is parallel to A D [21]. Let E F be a parallel to D C through the point E, and C F a parallel to A B through the point C; let also A B be produced to a point H of E F: then D C is equal to E F. Because O H is equal to D E [22], and consequently to A B [i], the line B H is equal to A O [viii], and consequently to O B [i]; that is, B is the middle point of O H. Because O is the middle point of D C, the point H is the middle point of E F [22 iii]. Because the straight line joining the points C and E passes through the point B [25 iii], it coincides with C B and B E [4]; thence C B E is a straight line: but A D is parallel to B E [*Dem.*]; therefore A D is parallel to C B, which W. T. B. D.

Corollary I. *If two straight lines bisect each other, the lines joining their opposite extremities are equal to each other.*

Corollary II. *The intersection of two straight lines bisecting each other, and the middle points of the lines joining their opposite extremities, are points of the same straight line.*

Corollary III. *If two equal straight lines bisect each other, two opposite lines joining their extremities are perpendicular to the two others.*

Corollary IV. *Two equal straight lines, the extremities of which* *e joined by lines, each equal to each of them, are parallel to each her.*

COROLLARY V. *If two straight lines be cut by two transversals which t each other, and if the parts of each of the latter determined by the ints of section be equal to each other, the two straight lines are rallel to each other.*

SECTION IV.

ON SYMMETRY.

DEFINITIONS.

22. Two points, or lines, are said to be *symmetric* to a raight line when, on the plane containing them being lded upon the latter as an axis, these points, or lines, coıcide with each other. Each of them is said to be the *ymmetric* of the other, and the straight line to which they re symmetric, is called their *line of symmetry*.

It is obvious that symmetric lines are equal to each other, and that the xtremities of symmetric lines are symmetric, also that points of symmetric ines equally distant from their symmetric extremities are symmetric. It is lso evident that straight lines are symmetric, and consequently equal to ach other, when their extremities are symmetric [3 i]; thus the line joinng symmetric points to a point of their line of symmetry are symmetric nd equal to each other. Again, it is evident that the intersection of the ymmetrics of two straight lines is the symmetric of that of the two ines.[1]

THEOREM 27.

Two points symmetric to a straight line have the same proection on the latter.

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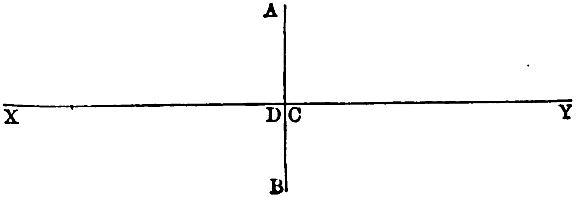
[1] Whenever in The Elements several figures are said to be symmetric, it is meant *ymmetric to the same straight line*, unless the contrary be stated.

Let A and B be two points symmetric to the indefinite straight line X Y.



Let A C and B D be two perpendiculars on X Y; then C is the projection of the point A on X Y, and D the projection of the point B.

Let the plane containing the figure be folded upon X Y; then the points A and B will coincide [*Hyp.*], and also the perpendiculars A C and B D [10]; therefore the points C and D coincide also, and are one and the same point: that is, the points A and B have the same projection on their line of symmetry, W. W. T. B. D.

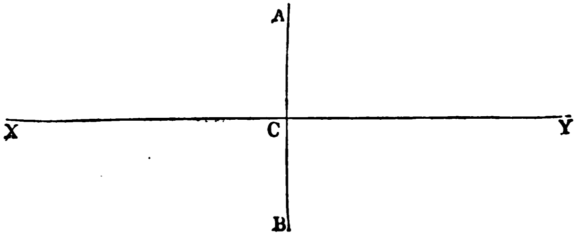
Corollary I. *Two symmetric lines have the same projection on the line of symmetry.*

Corollary II. *The perpendiculars from symmetric points to the line of symmetry, are symmetric.*

Corollary III. *Two points symmetric to a straight line are equally distant from it.*

Theorem 28.

A straight line is the middle perpendicular on the line joining two points symmetric thereto.



Let A and B be two points symmetric to X Y.

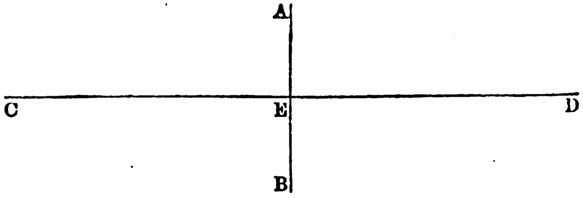
Let perpendiculars be drawn from A and B to X Y. They will have the same foot C [27], and will be equal to each other [27 iii]; also, C B will be the prolongation of A C [9 ii]: therefore X Y is the middle perpendicular on A B, which W. T. B. D.

COROLLARY. *The lines joining two points to their respective symmetrics are parallel to each other* (16).

THEOREM 29.

Conversely, *the extremities of a straight line are symmetric to its middle perpendicular.*

Let C D be the middle perpendicular on the straight line A B.



Let E be the middle point of A B, and let the plane containing the figure be folded upon C D; then E B will coincide with its prolongation E A [9 i], and the point B with the point A: therefore the points A and B are symmetric to C D, which W. T. B. D.

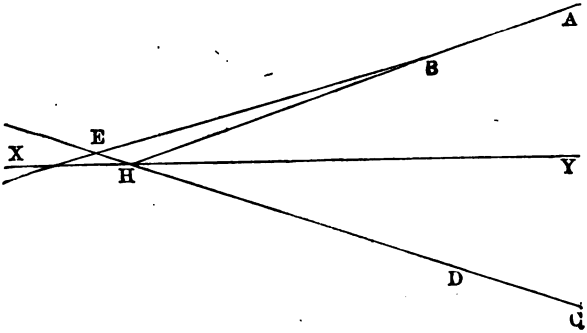
COROLLARY I. *Every point equally distant from two others, belongs to their line of symmetry.*

COROLLARY II. *If the extremities of two equal straight lines be joined by two other lines, each equal to the first, two adjoining lines are symmetric to the line joining their common extremity to the common extremity of the others.*

THEOREM 30.

If two symmetric straight lines, not parallel to each other, be indefinitely produced, they shall meet on the line of symmetry.

Let A B and C D be two straight lines oblique to each other, and symmetric to X Y.



If A B and C D be indefinitely produced, they will meet [18 i]. Let E be the point where the prolongation of A B meets that of C D, let H be the point where C D produced cut X Y, and let B H be joined. Now let the plane containing the figure be folded on X Y; then C H will coincide with A H. But C H is a straight line; therefore A H is also a straight line: that is, B H is the prolongation of A B. But B E is the prolongation of A B [*Const.*]; therefore B E coincides with B H [4], and the point E with the point H: that is, A B and C D produced cut X Y at the same point, W. W. T. B. D. Inversely

If two straight lines meet without another, no parts of them are symmetric to the latter.

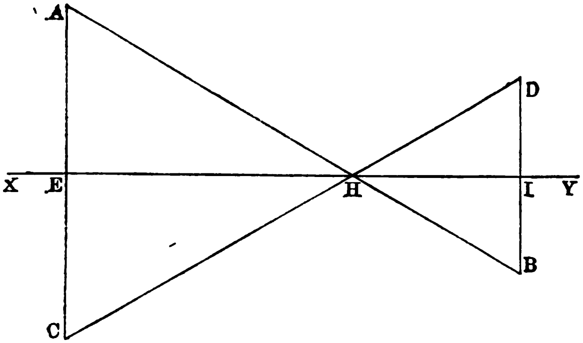
COROLLARY I. *If two symmetric straight lines cut each other, the intersection is on their line of symmetry.*

COROLLARY II. *If the perpendiculars through symmetric points of symmetric straight lines, not parallel to each other, be indefinitely produced, they shall meet on the line of symmetry*, and conversely.

THEOREM 31.

If a straight line cut the line of symmetry, its symmetric is a straight line.

Let A B be a straight line cutting the indefinite straight line X Y in H.



Let the points C and D be the symmetrics of the points A and B to X Y respectively; let A C and B D be joined, cutting X Y in E and I; also let the point H be joined to C and D. Then C H will be the symmetric of A H, and H D that of H B; that is, C H D will be the symmetric of the straight line A H B.

Let the part E A H B I of the figure turn upon X Y as an axis, the rest of the plane remaining fast, so that E A H turn on one side of the plane and H B I on the other side of it, until A H coincides with its symmetric C H. The whole of the straight line AB will again be in the same plane with the rest of the figure [4 iii], and H B will coincide with its symmetric H D. Now C H D coincides throughout with A H B; but A H D is a straight line [*Hyp.*]: therefore C H D also is a straight line, W. W. T. B. D.

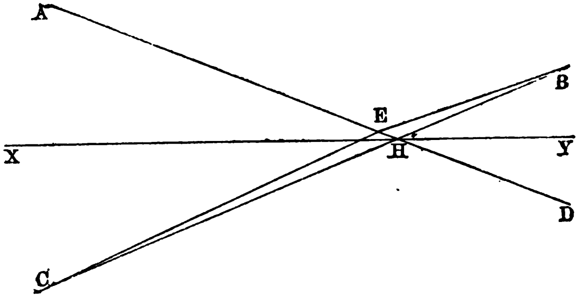
COROLLARY. *If two symmetric straight lines meet on the line of symmetry, their equal continuations beyond that line shall be symmetric thereto.*

Scholium. The symmetric of a straight line is a straight line.

THEOREM 32.

The straight lines which join each of two points to the symmetric of the other, cut each other on the line of symmetry.

Let the points C and D be the symmetrics of the points A and B to the indefinite straight line X Y respectively, and let A be joined to D and B to C by two straight lines, cutting each other in E.



Let the point H, where A D cuts the line of symmetry X Y, be joined to B and C; then C H B will be the symmetric of A D: therefore C H B is a straight line [31]. But C E B is a straight line [*Hyp.*], having the same extremities C and B; therefore C E B coincides with C H B [3 i], and the point E is one and the same with the point H: that is, the straight line joining B to C cuts A D on the line of symmetry, W. W. T. B. D. Inversely

If two straight lines meet without the line of symmetry, their extremities are not symmetric thereto.

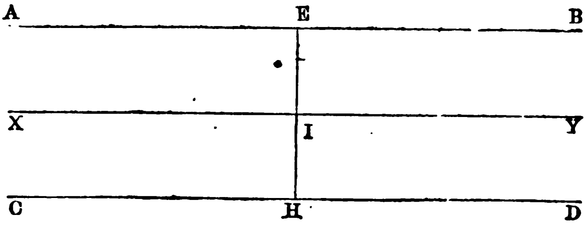
COROLLARY I. *The straight lines joining each of two points to the symmetric of the other, are symmetric.*

COROLLARY II. *If the line joining a point to the symmetric of the other cut the line of symmetry, the continuation of the line joining their intersection to the second point shall pass through the symmetric of the first.*

THEOREM 33.

Every point of each of two parallels equally distant from a third, has in the other a symmetric to the third parallel.

Let A B and C D be two parallels, equally distant from their parallel X Y.



Let E H be a perpendicular to A B cutting X Y in I; then E H is perpendicular to X Y and C D [16]. Because I E is equal to I H, [*Hyp.*], the line XY is the middle perpendicular on E H; therefore the points E and H are symmetric to X Y [29], which W. T. B. D.

COROLLARY I. *The intercepts of two parallels, equidistant from a third, between two perpendiculars to the latter, are symmetric to the same.*

COROLLARY II. *The intercepts of two parallels between two perpendiculars are symmetric to the middle line of the intercepts of the perpendiculars between them.*

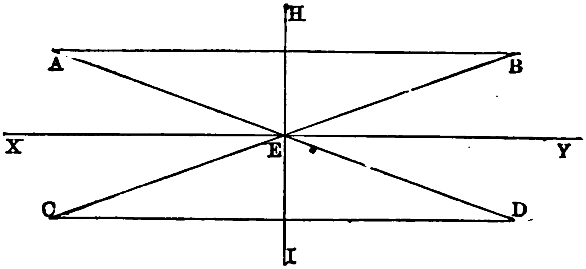
COROLLARY III. *If two equal straight lines bisect each other, two opposite lines joining their extremities, are symmetric to the middle line of the two others.*

COROLLARY IV. *Two symmetric parallels are parallel to their line of symmetry.*

THEOREM 34.

If two symmetric straight lines bisect each other, they are symmetric to the perpendicular on the line of symmetry drawn through their intersection.

Let the points C and D be the symmetrics of the points A and B to the indefinite straight line X Y respectively, and let the lines joining A to D and B to C bisect each other in E.



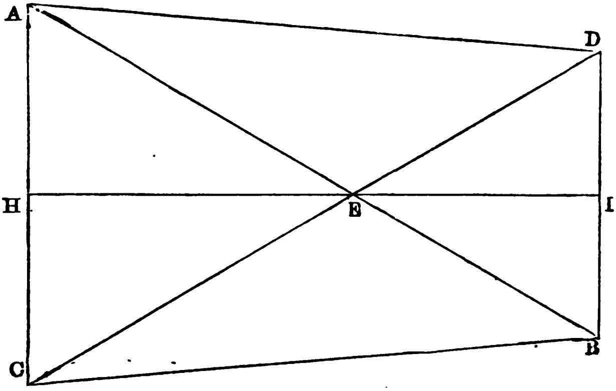
Then the point E is on the line of symmetry [32]. Let A B and C D be joined, and let H I be a perpendicular on X Y, drawn through the point E. Because A B and C D are parallel to each other [26], they are parallel to X Y [33 iv]; therefore H I is perpendicular to A B and C D [16]. Because A D and C B are symmetric to X Y, they are equal to each other, and A E is equal to B E [vii]; thence H I is the middle perpendicular on A B [15 iii], and the points A and B are symmetric to H I [29]; therefore A E and B E are symmetric to H I and also their equal continuations E C and E D [31 i], which W. T. B. D.

COROLLARY. *If two symmetric straight lines cut each other, and the plane containing them be folded upon the perpendicular on the line of symmetry drawn through their intersection, each of them coincides with the continuation of the other.*

THEOREM 35.

If two straight lines cut each other in unequal parts, and if the parts of one be equal to the parts of the other, each to each, the lines joining the extremities of unequal parts are symmetric to the middle line joining the extremities of equal parts.

Let A B and C D be two straight lines cutting each other in unequal parts at the point E; let E A be equal to E C, and E B equal to E D.



Let A D, C B, A C, and D B be joined, and let H I be a straight line through the point E, and the middle point H of A C; then H I is the middle perpendicular on A C [15 i], and the points A and C are symmetric to E H [29]; therefore A E and C E are symmetric to H I and also their equal continuations E B and E D [31 i]. Because the points B and D are symmetric to H I, the line H I is the middle perpendicular on B D [28]; therefore H I is the middle line of A C and B D. Now, the extremities of A D and C B are symmetric to H I; therefore A D and C B are symmetric to the same, W. W. T. B. D.

Corollary I. *If two straight lines cut each other in unequal parts, and if the parts of one be equal to the parts of the other, each to each, the lines joining the extremities of unequal parts are equal to each other; those joining the extremities of equal parts are parallel to each other; and the line joining the extremities of the greater parts is greater than that joining the extremities of the lesser.*

Corollary II. *If two straight lines bisect each other, they are symmetric to the middle line of the lines joining the extremities of the one to those of the other.*

CHAPTER II.

ON CURVES.

SECTION I.

ON CIRCUMFERENCES GENERALLY.

DEFINITIONS.

Elementary Geometry only deals with one kind of curve, called the *circumference of the circle.*

23. A *circumference of the circle,* or simply *circumference,* is a continuous line, all points of which are equally distant from an interior point, called *centre.*[1]

A circumference is accordingly the locus of all points of the same plane, equally distant from its centre. It is evident that if a straight line, the extremities of which are within the circumference, be indefinitely produced, it will cut the circumference in two points.

A circumference is generally designated by the letter at its centre; sometimes it is designated by letters at some of its points, and sometimes by the letter at the centre, together with letters at some of its points.

24. A straight line joining the centre to any point of the circumference is a *radius.*

It follows immediately from the definition of the circumference, that all radii are equal to one another.

25. An *arc* is a portion of a circumference; an arc smaller than half a circumference is an arc *of half the circle,* or a *semicircular* arc; an arc greater than half a circumference is an arc *of the circle,* or a *circular* arc.[2] Two arcs which together form a circumference are termed *oppo-*

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[1] A circumference of the circle is sometimes called a *circle.*

[2] In The Elements, the term *circular arc* has this restricted meaning only when used in opposition to that of *semicircular arc;* otherwise it means *any* arc.

site. The straight line which joins the extremities of an arc is its *chord:* the chord *subtends* the arc.

It is evident that each chord in a circumference subtends two opposite arcs.

An arc is designated by the letters at its extremities.[1] If several arcs have the same extremities, each one is designated by three letters: two at its extremities, and a third at some other of its points.

If a circumference be supposed to be divided into any number of equal parts, each of the arcs may be used as a unit to measure the whole circumference or a part of it. The unit generally adopted is the *degree*, which is an arc equal to the *three hundred and sixtieth part of the circumference:* thus the whole circumference contains 360 degrees; the half circumference, 180 degrees; and the quarter of the circumference, 90 degrees. The sixtieth part of a degree is called a *minute;* the sixtieth part of a minute is called a *second.*[2]

It is obvious that a chord which, in a circumference, subtends an arc of *one* degree, also subtends an arc of 359 degrees; and that, in general, if the number of degrees of one of the arcs subtended by a chord be known, the value of the other arc subtended by the same chord will be the difference between that number and 360 degrees.

26. A chord which passes through the centre of the circumference, is a *diameter:* a circumference is *described* on its diameter.

It is evident that a diameter is composed of two radii, one of which is the continuation of the other beyond the centre, and that its middle point is the centre of the circumference; also that all diameters of a circumference are equal to one another.

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[1] When one of two unequal arcs formed by the same chord in the same circumference is designated by the letters at its extremities only, the semicircular arc only is meant, unless the contrary be stated.

[2] With numbers written in figures, the marks °, ′, ″, are generally used for denoting the degrees, minutes and seconds; the fractional parts of a degree less than a second are expressed in decimal parts of the second. Thus, $14^\circ\ 7'\ 39{\cdot}5''$ stands for 14 degrees, 7 minutes and 39½ seconds.

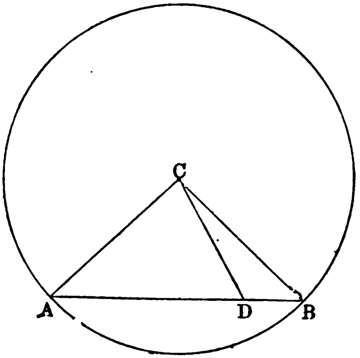
In several modern French works, the fourth part of the circumference is supposed to be divided into 100 equal parts, called *grades;* the hundredth part of a grade is called a *minute*, and the hundredth part of a minute is called a *second.* In this system, the above number would be written $14^{g}.07305$.

POSTULATE IX.

The elements of a circumference are equal to one another.

THEOREM 36.

No part of a chord is without the circumference.



Let AB be a chord subtending an arc of the circumference C.

Let the centre C be joined to the extremities of the chord and to any point D of the chord between A and B; then C A and C B are two radii [*Def.* 24]. Because C A and C B are equal to each other [*Def.* 23], the line C D is shorter than C A or C B [13 iii]; that is, less than a radius: therefore the point D is within the circumference. The same construction and reasoning would prove that every point of A B between its extremities is joined to the centre by a line shorter than a radius: therefore no point of A B is without the circumference C, which W. T. B. D.

COROLLARY I. *No part of the straight line joining two points within the circumference, is without the circumference.*

COROLLARY II. *Every point of the continuation of a chord is without the circumference.*

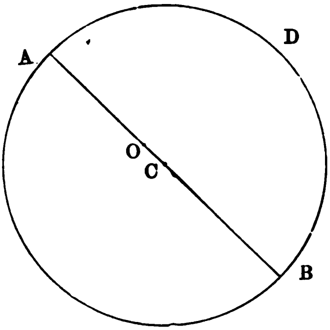
COROLLARY III. *The extremities of a chord are farther from the centre than any other of its points.*

THEOREM 37.

A circumference has but one centre.

Let C and O be centres to the circumference A B D.

Let O C be joined and produced to the points A and B of the circumference. Because the point C is a centre to the circumference A B D, the line A C is equal to B C; that is, A C is equal to half of A B. Because the point O is a centre of the same circumference, A O is equal to B O; that is, A O is equal to half of A B. But A C is equal to half of A B; therefore A O is equal to A C [i]: that is, the points C and O coincide with each other. Thus, the centres C and O are not distinct points, but one and the same centre: therefore the circumference A B D has only one centre, W. W. T. B. D.



Corollary I. *Two equal circumferences which coincide throughout, have the same centre.*

Corollary II. *Two equal circumferences have equal radii, and equal diameters.*

Theorem 38.

Two circumferences with equal radii, are equal to each other.

For if the two circumferences be applied to each other, so that their centres and radii coincide, they will coincide; otherwise, there would be points of one of them unequally distant from the centre, which is impossible [*Def.* 23].

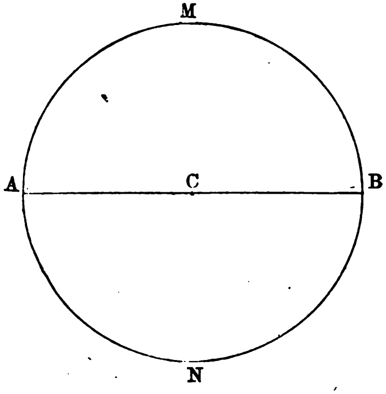
Corollary I. *Two equal circumferences coincide throughout with each other when their centres coincide.*

Corollary II. *Arcs of equal circumferences, which have the same extremities, are equal to each other.*

Scholium. A circumference is determined when its centre and radius are known.

THEOREM 39.

Every diameter divides the circumference into two equal parts.



Let C be the centre of the circumference A M B N, and A B a chord through that centre, dividing the circumference into two arcs A M B and A N B.

Let the plane containing the circumference be folded upon A B. Every point of the arc A M B will coincide with A N B, otherwise there would be, in the circumference, points not equally distant from the centre: which is impossible [*Def.* 23]; therefore the arc A M B is equal to the arc A N B, which W. T. B. D. Inversely

Any chord which divides the circumference into two unequal parts does not pass through the centre.

COROLLARY I. *If a circumference be folded upon its diameter, the arcs of the two halves which have the same extremities are equal to each other.*

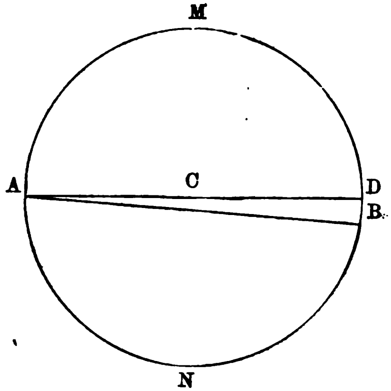
COROLLARY II. *Every diameter divides the circumference into two parts symmetric to itself.*

THEOREM 40.

Conversely, *a chord which divides the circumference into equal parts, is a diameter.*

Let A B be a chord dividing the circumference A M B N into equal parts, A M B and A N B.

Let C be the centre of the circumference A M B N, and let A C be joined and produced to a point D of the circumference. The chord A D will be a diameter [*Def.* 26]; therefore the arc A M D is half of the circumference [39]. But the arc A M B is half of the circumference [*Hyp.*]; therefore the arcs A M B and A M D are equal to each other: that is, the points B and D coincide with each other. Because the chords A B and A D have the same extremities, they coincide with each other and form one and the same chord [3 i]: but A D is a diameter; therefore the chord A B is a diameter, W. W. T. B. D. Inversely

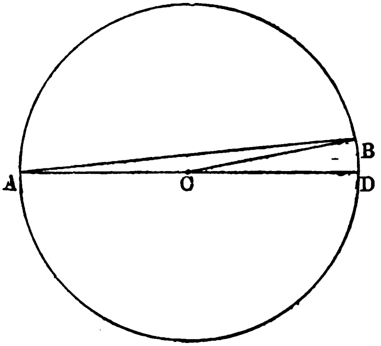


Any chord not passing hrough the centre divides the ircumference into unequal parts.

COROLLARY. *If a circumference be divided into two arcs, the straight ine joining their middle points is a diameter.*

THEOREM 41.

Any chord, not passing through the centre, is less than a diameter.



Let A B be a chord not passing through the centre C.

Let A C and B C be joined, nd let A C be produced to he circumference in D; then A D is a diameter [*Def.* 26]. The straight line A B is less than A C and C B together [3 i]; now the radii C B and C D are equal to each ther [*Def.* 23]; therefore

A B is less than A C and C D together, or A D: that is, A B is less than a diameter, W. W. T. B. D. Inversely

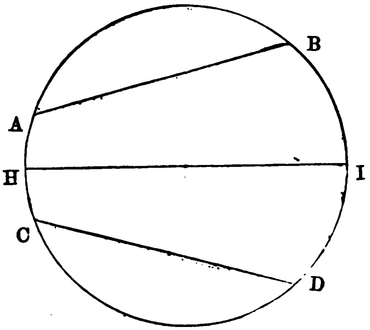
A chord equal to a diameter is a diameter.

COROLLARY I. *The diameter is the greatest chord that can be drawn in a circumference,* and conversely.

COROLLARY II. *The radius measures the shortest distance from one of its points to the circumference.*

THEOREM 42.

Equal semicircular arcs of a circumference are subtended by equal chords.



Let A B and C D be two equal arcs in the same circumference.

Through the middle point H of the arc A C let there be a diameter H I, and let the plane containing the circumference be folded upon H I; then the two arcs H A B I and H C D I will coincide with each other [39]. The two arcs H A and H C being equal to each other [*Const.*], the points A and C will coincide with each other; and, the arcs A B and C D being also equal to each other [*Hyp.*], the points B and D will also coincide with each other. Now, the chords A B and C D have the same extremities; therefore they are equal to each other [31], W. W. T. B. D. Inversely

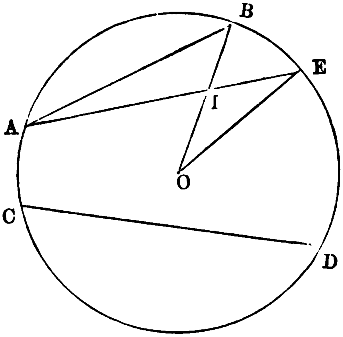
Unequal chords in a circumference subtend unequal semicircular arcs.

COROLLARY I. *Equal circular arcs of a circumference are subtended by equal chords.*

COROLLARY II. *Equal arcs in equal circumferences are subtended by equal chords.*

THEOREM 43.

Unequal arcs of a circumference are subtended by unequal chords.



In the circumference O, let there be an arc C D greater than another arc A B.

Let the arc A B E be equal to the arc C D; let A E be joined, and also the centre O, to the points B and E, and let I be the point where the radius O B cuts the chord A E; then the straight line O E is less than O I and I E together [6 i]. Because O E and O B are two radii, they are equal to each other; thence O B is also less than O I and I E together. Now, the straight line A B is less than A I and I B together; therefore A B and O B are together less than A I, I B, O I and I E, or less than A E and O B [v]: therefore A B is less than A E [iii]. But the chords A E and C D are equal to each other [42]; therefore A B is less than C D, which W. T. B. D. Inversely

Equal chords in a circumference subtend equal arcs (Conv. of 42).

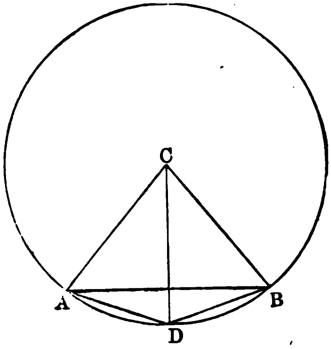
COROLLARY I. *Equal chords in equal circumferences subtend equal arcs.*

COROLLARY II. *In the same circumference, or in equal circumferences, a greater semicircular or a smaller circular, arc is subtended by a greater chord,* and conversely.

THEOREM 44.

The radius perpendicular to a chord bisects the chord and its arc.

Let C D be a radius perpendicular to the chord A B.



Let both the points C and D be joined to A and B : then, because the radii A C and B C are equal to each other [*Def.* 23, the perpendicular C D bisects the line A B [15 iii]. Now, because C D is the middle perpendicular on A B, the chord A D is equal to the chord D B [14]; therefore the arc A D is equal to the arc D B [42], which W. T. B. D. Inversely

A perpendicular to a chord dividing it, or its arc, in unequal parts, does not pass through the centre, and

The radius dividing a chord or its arc into unequal parts is not perpendicular to the chord.

Corollary I. *The centre of a circumference and the middle points of an arc and its chord, are points of the same straight line perpendicular to the chord.*

Corollary II. *The centre of a circumference and the middle points of parallel chords are points of the same diameter perpendicular to the chords.*

Corollary III. *Two diameters perpendicular to one another divide the circumference into four parts.*

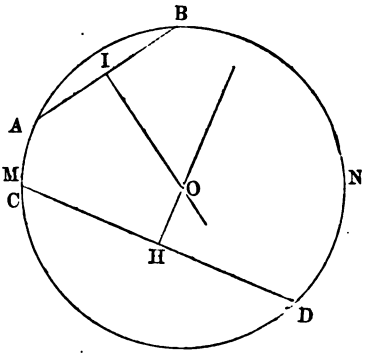
Corollary IV. *Two chords bisecting each other have their intersection at the centre.*

Scholium. A straight line answering any two of the conditions of the Corollaries I or II, will answer the two others [4].

Theorem 45.

The middle perpendiculars on two oblique chords cut each other at the centre of the circumference.

Let A B and C D be two oblique chords, and let I O and H O be heir middle perpendiculars.



The perpendiculars I O and I O will meet in a point O 18 ii]; and the centre of the ircumference, being in both of hem [44 i], is at their intersecion; therefore the middle perendiculars on A B and C D will ut each other at the centre, W. W.T.B.D.

Corollary I. *Through three oints not in a straight line there ay always be drawn a circumerence.*

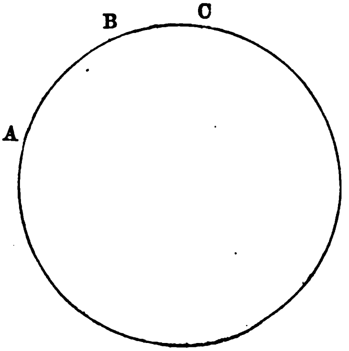
Corollary II. *Through three points there cannot be more than one ircumference.*

Corollary III. *A point equally distant from three points of the cirumference is the centre thereof.*

Scholium. A circumference is determined when three of its points re known.

Theorem 46.

Two arcs which have three points in common are parts of equal circumferences.



Let two arcs have the three points A, B and C in common.

Let the circumferences of the two arcs be completed. Through the three points A, B and C, there can be but one circumference [45 ii]; thence the two circumferences A B C oincide throughout their exension, and *are not distinct* from each other; therefore they are equal to each other, W. W. T. B. D.

COROLLARY I. *If two arcs coincide with each other, their centres coincide.*

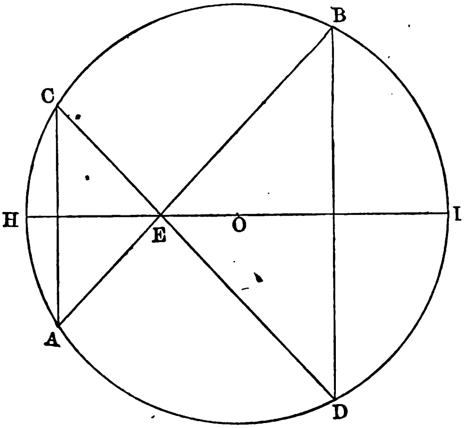
COROLLARY II. *Equal arcs are parts of equal circumferences.*

COROLLARY III. *If two arcs having the same extremities have a third point in common, they coincide throughout with each other.*

THEOREM 47.

If two equal chords of a circumference cut each other, the portions of the one are equal to those of the other, each to each.

Let two equal chords A B and C D of the circumference O cut each other in the point E.



Let A C and B D be joined, and, through the middle point H of the arc A C, let there be a diameter H I; then H I is the middle perpendicular to A C [44 i]. Because the arcs A C B and C A D are equal to each other [43], and the arc A C is common to both, the arc A D is equal to the arc C B [viii]; thence the point H is the middle point of the arc D H B [iv]; therefore H I is the middle perpendicular to B D [44 i]. Because the points A and C are symmetric to H I [29], and also the points B and D, the lines A B and C D cut each other on H I [32]; thence A E and C E are symmetric and also E B and E D; therefore A E is equal to C E, and ıual to E D, which W. T. B. D.

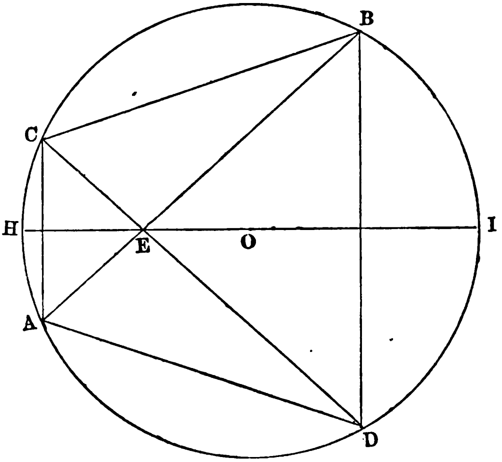
LARY. *The intersection of two equal chords of a circumference le points of the chords joining their extremities and of the arcs d by the latter, are, with the centre of the circumference, points me straight line.*

THEOREM 48.

'ersely, *if two straight lines cut each other, and the s of the one be equal to those of the other, each to each, tremities are points of the same circumference.*

.B and C D be two straight lines cutting each other in E, A E and E B be respectively equal to C E and E D.

, circumfer- C A D be hrough the C, A and A C, B D, ıd C B be and let H I rd through ldle points and D B; C is a chord circumfer- A D. Be- C B and symmetric [35], the A and C are symmetric to H I; thence H I is the middle licular to the chord A C [28]: therefore H I is a diameter circumference C A D. Let the figure be folded upon H I;

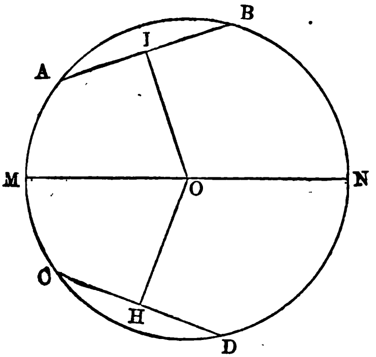


then the semicircumference H C I will coincide with the semicircumference H A I [39]. Because the points B and D are symmetric to H I, the points B and D will coincide with each other; therefore the point B is a point of the circumference C A D, which W. T. B. D.

Corollary. *The intersection of equal straight lines bisecting each other is the centre of the circumference passing through their extremities.*

Theorem 49.

Equal chords in a circumference are equally distant from the centre.



Let A B and C D be two equal chords, and let O I and O H be perpendiculars on them from the centre O.

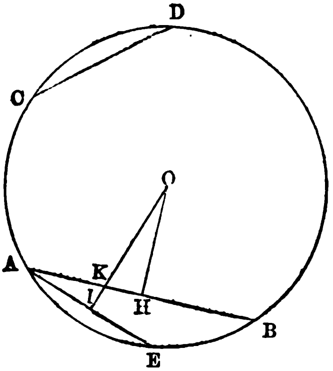
Let M N be a diameter through the middle point M of the arc A C, and let the plane containing the circumference be folded upon M N: then the semicircumference M A B N will coincide with M C D N [39]; also the points A and B, with the points C and D respectively, and the chord A B with the chord C D. Because the centre O has remained unmoved, the perpendicular O I will coincide with O H [10] and the point I with the point H. Now, the perpendiculars O I and O H, having the same extremities, are equal to each other; therefore the chords A B and C D are equally distant from the centre [11], W.W. T. B. D. Inversely

Chords not equally distant from the centre of a circumference are unequal.

COROLLARY. *Equal arcs in equal circumferences are equally distant from the centre.*

THEOREM 50.

Unequal chords in a circumference are not equally distant from the centre.



Let A B and C D be two chords of the same circumference O, and let A B be greater than C D.

Let the arc A E be equal to the arc C D, and let A E be joined; then the chord A E is equal to C D [42]. Let also O H and O I be perpendiculars on A B and A E, and let K be the point where O I cuts A B. Because O H is perpendicular on A B, it is less than the oblique O K [11], and still less than O I; therefore the chord A B is nearer to the centre than A E. But, because the chords C D and A E are equal to each other, they are equally distant from the centre [49]; therefore the chord A B is nearer to the centre than C D, which W. T. B. D. Inversely

Chords equally distant from the centre of a circumference are equal to one another. (Conv. of 49.)

COROLLARY I. *Chords equally distant from the centre in equal circumferences are equal to one another.*

COROLLARY II. *In the same circumference, or in equal circumferences, a greater chord is nearer to the centre.*

SECTION II.

ON TANGENTS.

DEFINITIONS.

27. A circumference and a straight line which have one point in common and no more, are *tangent* to one another; the straight line is a *tangent*, and the point it has in common with the circumference is called the point of *tangence*, or of *contact*.

It is evident that a tangent is infinite on each side of the point of tangence. However, if there be more than one tangent to a circumference from a point without it, the length of each of them is reckoned from that point to its point of tangence.

28. A circumference and a straight line which have more than one point in common, are *secant* to one another: the straight line is a *secant*, and the points it has in common with the circumference are called its points of *section;* the part thereof intercepted by the circumference is its *chord*. A *centre* secant is that which passes through the centre of the circumference.

It is obvious that a straight line which meets a circumference is either a tangent or a secant thereto.

A secant is necessarily infinite on each side of the circumference to which it is secant. However, if several secants be drawn to a circumference from a point without it, the length of each of them is reckoned from that point to the farthest point of section; the part of each of them contained between the same point and the circumference is its *outer* part.

29. If two tangents, or two equal secants, be drawn to a circumference from a point without it, that of the two arcs intercepted between them which is nearest to their common point is *convex* to that point; the other arc is *concave* to the same point.

THEOREM 51.

A straight line cannot meet a circumference in more than
points.

or, otherwise, the centre could be joined to the straight line by re than two equal straight lines, which is impossible [13 ii].

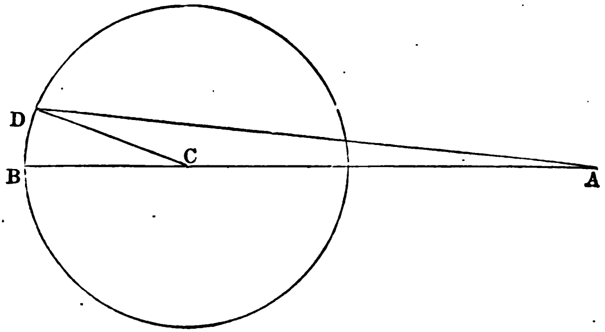
OROLLARY. *Through three points in a straight line there cannot be wn a circumference.*

cholium. There is no common measure between a straight line a circumference.

THEOREM 52.

f all straight lines that can be drawn to a circumference m a point without it, the greatest is that which passes ough the centre.

et the point A without the circumference C, be joined to the t B thereof through the centre, and let any other point D of the umference be joined to A and to the centre.



he straight line AD is less than AC and CD together [6 i]; the radii CB and CD are equal to each other: therefore AD is less than A C and C B together; that is, less than A B, which W. T. B. D.

COROLLARY I. Conversely, *if a straight line be the greatest that can be drawn to a circumference from a point without it, it passes through the centre.*

COROLLARY II. *Of all straight lines that can be drawn to a circumference from a point without it, the least is that the continuation of which passes through the centre,* and conversely.

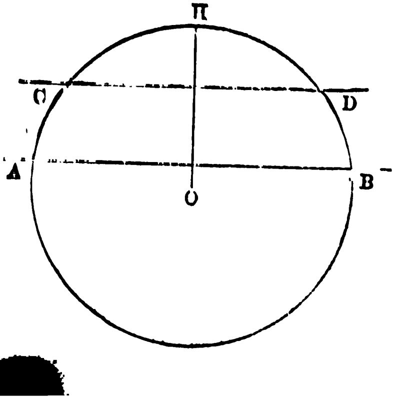
COROLLARY III. *If two secants drawn to a circumference from the same point without it be equally distant from the centre, they are equal to each other,* and conversely.

COROLLARY IV. *From a point without a circumference, there cannot be drawn more than two equal secants to it.*

COROLLARY V. *If two equal secants be drawn to a circumference from a point without it, that point, the middle points of the convex and concave arcs and of their chords, are, with the centre of the circumference, points of the same straight line* (47 i).

THEOREM 53.

The arcs of a circumference intercepted between two parallel secants are equal to each other.



Let A B and C D be two parallels intercepting the arcs A C and B D in the circumference O.

From the centre let O H be a radius perpendicular on A B and consequently on C D. The arcs A H and B H are equal to each other [44]; also the arcs C H and D H; therefore the difference of the arcs A H and C H is equal to the difference of the arcs B H and D H [iv]; that is, the arc A C is equal to the arc B D, which W. T. B. D. Inversely

If two arcs be unequal, the secants passing through their extremities are not parallel to each other.

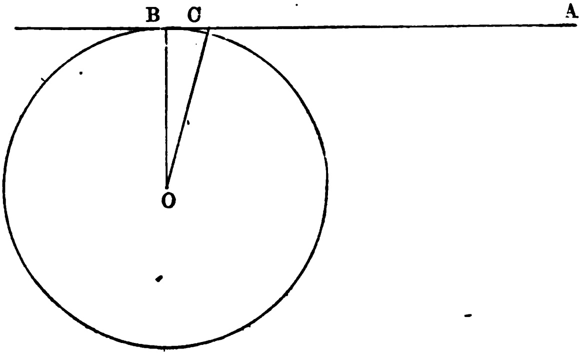
COROLLARY I. Conversely, *if two arcs of a circumference be equal to each other, the chords joining their extremities without crossing each other, are parallel to each other.*

COROLLARY II. *If two equal secants be drawn to a circumference from a point without it, the chords of the arcs intercepted between them are parallel to each other.*

THEOREM 54.

A straight line perpendicular at the extremity of a radius is tangent to the circumference.

Let A B be a straight line perpendicular at the extremity of the radius O B.



Let any point C of A B, other than B, be joined to the centre; then O C will be an oblique to A B [10] greater than O B [11]: but O B is a radius; therefore the point C is outside the circumference O. As the same construction and reasoning could be applied to any other point of A B but B, it is evident that every point of A B but B is outside the circumference O; therefore the straight line A B has only one point in common with the circumference O: that is, A B is tangent to that circumference, W. W. T. B. D. Inversely

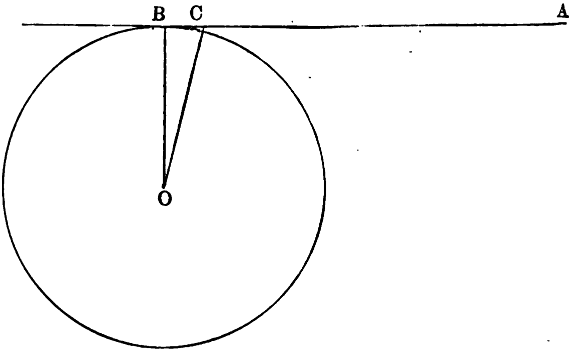
A secant is oblique to the radius terminating at a point of section.

COROLLARY. *From a point without a circumference there may be two tangents thereto.*

THEOREM 55.

Conversely, *a tangent to a circumference is perpendicular to the radius terminating at the point of tangence.*

Let A B be a tangent to the circumference O at the point B.



Let the point of tangence be joined to the centre O, then O B, being a radius, will be shorter than O C joining the centre to any other point of A B, all these points being outside the circumference [*Def.* 27]. Then O B is the shortest of all straight lines joining the point O to A B; therefore O B is perpendicular to A B [11 i], which W. T. B. D. Inversely

A straight line oblique at the extremity of a radius, is secant to the circumference.

OROLLARY I. *Through a point of a circumference, there cannot be e than one tangent to the circumference* (9).

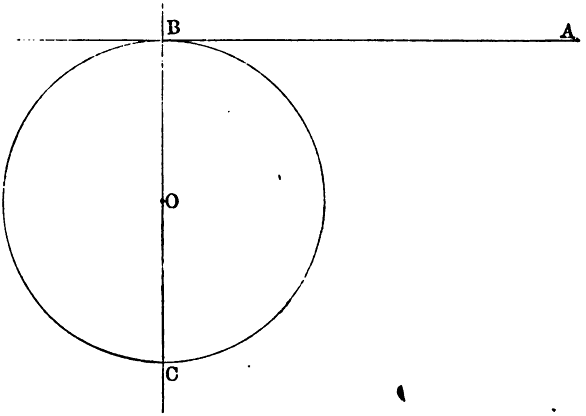
OROLLARY II. *The tangents at the extremities of a diameter, are allel to each other*, and conversely.

OROLLARY III. *The arcs of a circumference intercepted between a gent and a parallel secant, or tangent, are equal to each other.*

THEOREM 56.

The perpendicular raised on a tangent through the point of gence passes through the centre of the circumference.

et A B be a tangent at the point B to the circumference O, and let be a perpendicular on A B at the same point.



f the centre O be joined to the point of tangence, the radius O B l be perpendicular on A B [55], and consequently will coincide h B C [9]; therefore, B C will pass through the centre, W. W. .D. Inversely

If the perpendicular drawn through the point of a straight line in common with a circumference does not pass through the centre, the straight line is secant to the circumference.

COROLLARY I. *The chord joining the points of tangence of two tangents drawn to a circumference from a point without it, is less than the diameter.*

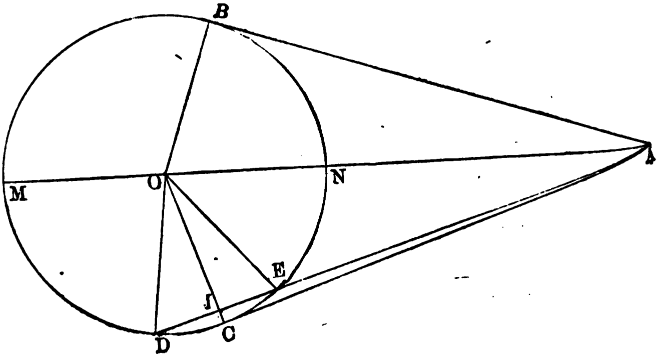
COROLLARY II. *The arc convex to a point from which two tangents, or two equal secants, are drawn to a circumference, is less than the arc concave to the same points.*

Scholium. The chord perpendicular to a tangent through the point of tangence is a diameter.

THEOREM 57.

Two tangents to a circumference drawn from a point without it are equal to each other.

Let A B and A C be two tangents from the point A to the circumference O.



Let O B and O C be two radii terminating at the points of tangence, and A N M a secant passing through the centre O. Let

plane containing the figure be folded upon A M. Because M N
diameter [*Def.* 26], the arc M B N will coincide with the arc
C N [39]. Also the point B will coincide with the point C; for
erwise, the radius O B would take a position O D, or O E, on either
e of O C; the tangent A B would become A D, or A E, and cut the
ius O C in a point I, and O C, being perpendicular on A C [55],
ld be an oblique on A D [10], whilst O D, or O E, would be perpen-
ular on A D [55]. The oblique O I on A D, which is less than a
lius, would then be less than the radii perpendicular O D, or O E,
m the same point, which is impossible [11]; and the further D or
should be removed from C, the less O I would become: therefore
or E, coincide with C and A B with A C; that is, A B is equal to
C, which W. T. B. D. Inversely

*If one of two unequal straight lines drawn to a circumference from a
int without it be a tangent, the other is a secant.*

COROLLARY I. *From a point without a circumference there cannot be
re than one tangent to it on the same side of the centre secant from
same point.*

COROLLARY II. *The tangent is less than any secant drawn from the
ne point without the circumference, and greater than its outer part.*

COROLLARY III. *From a point without a circumference there cannot
more than two equal straight lines drawn thereto.*

COROLLARY IV. *The tangents drawn to a circumference from a point
hout it are symmetric to the centre secant from the same point.*

COROLLARY V. *The straight line drawn through the centre from a
nt without the circumference, bisects the convex and concave arcs to
t point.*

SECTION III.

ON TANGENT CIRCUMFERENCES.

DEFINITIONS.

30. Two circumferences which have no point in common are *internal to one another*, or simply *internal*, when no part of the one is outside the other; they are *external to one another*, or simply *external*, when no part of the one is within the other.

31. Two circumferences are *concentric to each other*, or simply *concentric*, when they have the same centre, and *excentric* when they have distinct centres.

32. Two circumferences which have one point in common, and no more, are *tangent to one another*, or simply *tangent*; their common point is the point *of tangence*, or *of contact*.

Two circumferences may be *internally*, or *externally*, tangent to one another.

33. Two distinct circumferences which have more than one point in common are *secant to one another*, or simply *secant*; the points in common are their points *of section*, and the straight line joining them is their *common chord*.

Two secant circumferences cannot have more than two points of section since they would then coincide.

It is clear that two circumferences can have but five different positions relatively to one another: they may be *internal*, *internally tangent*, *secant*, *externally tangent*, or *external* to one another.

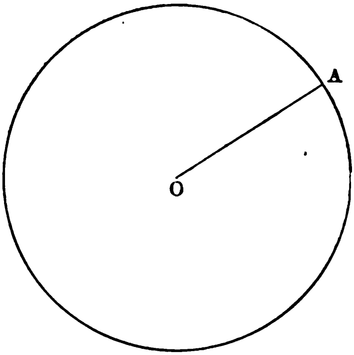
34. The straight line which joins the centres of two circumferences, is their *centre line*.

POSTULATE X.

ıy point may be the centre of a circumference of a given us.

THEOREM 58.

' two concentric circumferences have one point in common, coincide throughout with each other.



t two circumferences have
ame centre O and the point
common.
t O A be joined: then O A
ıeir common radius [*Def.*
Because the two circumfer-
ɜ have the same radius,
are equal to each other
: now, they have the same
·e O; therefore they coin-
;hroughout with each other
|, W.W.T.B.D. Inversely
ɔo concentric circumferences which do not coincide throughout with other, have no point in common.

ROLLARY I. *If two circumferences have no less than one point and ɔre than two points in common, they have not the same centre.*

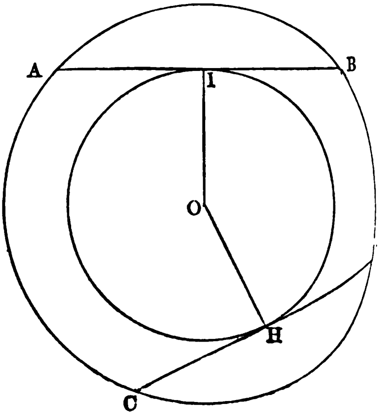
ROLLARY II. *The distance between two concentric circumferences ɩal to the difference of their radii.*

ıolium. Two concentric circumferences are throughout equally ıt from each other (*Def.* 23).

THEOREM 59

two concentric circumferences, all chords of the greater ı are tangent to the lesser, are equal to one another.

Let two circumferences A D and I H have the same centre O, and let the chords A B and C D of the greater be tangent in I and H to the lesser.



Let O I and O H be joined. Because O I and O H are radii of the same circumference, they are equal to each other, and because they terminate at the points of tangence I and H, they are respectively perpendicular to A B and C D [55]; thence the chords A B and C D are equally distant from the centre O : therefore they are equal to each other [50]. The same construction and reasoning would pro that all other chords of the greater circumference, tangent to t lesser, are equal to A B and consequently equal to one another, whi W. T. B. D.

Corollary I. *In two concentric circumferences, all chords of greater which are tangent to the lesser, are bisected at the point tangence.*

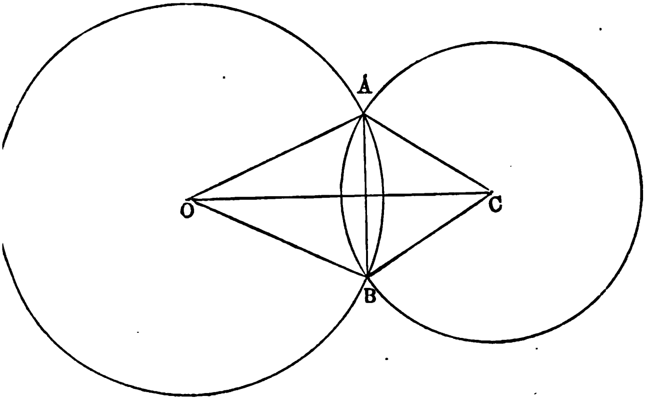
Corollary II. *The intercepts of a secant between two concentric cumferences are equal to each other.*

Corollary III. *The locus of the middle points of all equal ch of a circumference is a concentric circumference tangent to al them.*

Theorem 60.

In two secant circumferences, the centre line is the mi perpendicular of the common chord.

et O and C be two circumferences secant to each other in A and let O C be their centre line, and A B their common chord.



et the centres O and C be joined to A and B. Because the radii A and O B are equal to each other, and because the radii C A and are also equal to each other, the line O C has two points, O and C ally distant from A and B; therefore O C is the middle perpenular on A B [15 ii], which W. T. B. D.

OROLLARY I. *The points of section of two secant circumferences are metric to the centre line.*

OROLLARY II. *If the common chord of two secant circumferences ct their centre line, they are equal to each other,* and consely.

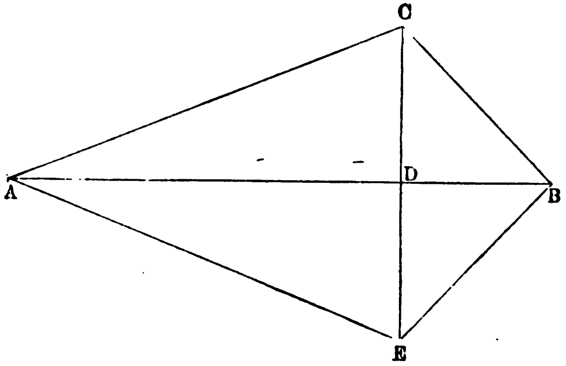
OROLLARY III. *If the common chord and the centre line of two l secant circumferences be equal to each other, each of these lines reater than the radius.*

OROLLARY IV. *If the common chord of two equal secant circumnces be equal to their radius, their centre line is greater than their us.*

THEOREM 61.

If two circumferences have one point in common without their centre line, they are secant to one another.

Let A B be the centre line of two circumferences, and let C be a point in common to both of them, so that A C is a radius of the circumference A, and B C a radius of the circumference B.



Let C D be a perpendicular on A B, and let it be produced to a point E symmetric of C. The line A B will then be the middle perpendicular of C E [28]; A E will be equal to A C [14], and B E to B C: thence the two circumferences drawn from the centres A and B with the radii A C and B C, will both pass through the point E; therefore the circumferences A and B are secant in C and E, which W. T. B. D. Inversely

The point of tangence of two tangent circumferences is in their centre line.

COROLLARY I. *If two circumferences have a point in common in their centre line, they are tangent to each other.*

COROLLARY II. *If any number of circumferences be tangent to one another in the same point, that point and all the centres are in the same straight line.*

THEOREM 62.

If two circumferences be externally tangent to one another, heir centre line is equal to the sum of their radii.

For the point of tangence is in their centre line [61]: therefore he centre line is composed of the radii of both circumferences.

COROLLARY I. Conversely, *if the centre line of two circumferences e equal to the sum of their radii, those circumferences are externally angent.*

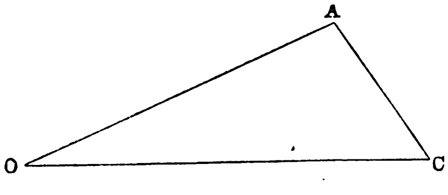
COROLLARY II. *If two circumferences be internally tangent, their entre line is equal to the difference of their radii,* and conversely.

Scholium. The centre line of two external circumferences is greater han the sum of their radii, and that of two internal circumferences s less than the difference of their radii.

THEOREM 63.

If two circumferences be secant to one another, their centre ne is less than the sum of their radii and greater than their ifference.

Let O C be the centre line of two circumferences secant to one nother, and let A be one of their points of section.



The circumferences being secant to one another, the point of ection A is without their centre line [61 i]. Let the point A be joined to the centres O and C; then O A and A C will be the radii of the two circumferences.

Firstly, the straight line O C is less than O A and A C together [6 i]; that is, the centre line is less than the sum of the radii.

Secondly, the straight line O A is less than O C and A C together [6 i]. Then, the difference between O A and A C is also less than O C and A C together minus A C [ii], or less than O C: that is, the centre line is greater than the difference of the radii.

Therefore, the centre line O C is less than the sum of the radii O A and A C, and greater than their difference, W. W. T. B. D.

THEOREM 64.

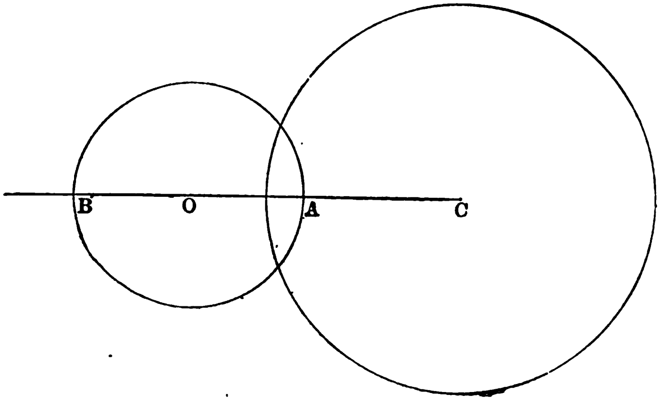
Conversely, *if the centre line of two circumferences be less than the sum, and greater than the difference, of their radii, those circumferences are secant to one another.*

Let the centre line O C of two circumferences be less than the sum, and greater than the difference, of their radii, and let A and B be the points of it and of its continuation where it is cut by the circumference O.

Should the circumference C cut the centre line between C and A, the two circumferences would be external to one another, and their centre line greater than the sum of their radii, which is impossible [*Hyp.*].

Should the circumference C cut O C in A, the two circumferences would be externally tangent to one another [61 i], and their centre line equal to the sum of their radii [62], which is impossible [*Hyp.*].

Should the circumference C cut the continuation of O C in B, the two circumferences would be internally tangent to one another [61 i], and their centre line equal to the difference of their radii [62 ii], which is impossible [*Hyp.*].



Finally, should the circumference C cut the continuation of O C beyond B, the two circumferences would be internal to one another and their centre line less than the difference of their radii, which is impossible [*Hyp.*].

Thus, the circumference C does not cut A B in its extremities, neither in its continuation between C and A, nor beyond B; therefore it cuts A B: that is, the circumferences O and C are secant to one another [36], W. W. T. B. D.

COROLLARY I. *If the centre line of two circumferences be greater than the sum of their radii, those circumferences are external to one another.*

COROLLARY II. *If the centre line of two circumferences be less than the difference of their radii, those circumferences are internal to one another.*

SECTION IV.

ON COMMON TANGENTS.

DEFINITIONS.

35. A *common* tangent is a straight line tangent to two circumferences; an *intangent* is a common tangent which crosses the centre line; an *extangent* is a common tangent which does not cross the centre line. The points of *intangence,* or *extangence,* of two circumferences are the points of tangence of their intangents, or extangents.

The length of a common tangent is its length between its points of tangence.

36. A *common* secant is a straight line secant to two circumferences.

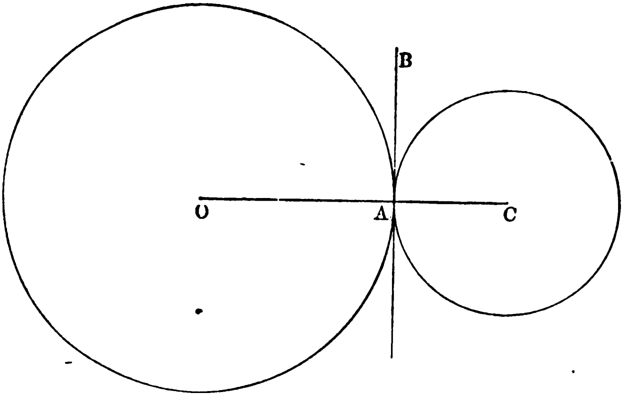
The length of a common secant is its length between its points of section which are the furthest from one another; its *outer* part is that intercepted between the two circumferences.

THEOREM 65.

If two circumferences be tangent to each other, the straight line tangent to one of them in their point of tangence is tangent to the other.

Let two circumferences O and C be tangent to each other in A, and let the straight line A B be tangent to the circumference O in the same point A.

Let the centre line O C be joined: it will pass through the point A [61]. Now B A is perpendicular on O A [55], and consequently on its continuation A C; but A C is the radius of the circumference therefore B A is tangent to the circumference C [54], which T.B.D. Inversely



If two circumferences be tangent to each other, the secant to one of em in their point of tangence, is secant to the other.

Corollary I. *There cannot be more than one common tangent to two rcumferences internally tangent to each other, nor more than one tangent to two circumferences externally tangent to each other.*

Corollary II. *The intangent of two equal tangent circumferences middle perpendicular on the centre line.*

Corollary III. *The circumference drawn from a point of the intanent of two externally tangent circumferences through their point of angence, is tangent in that point to their centre line.*

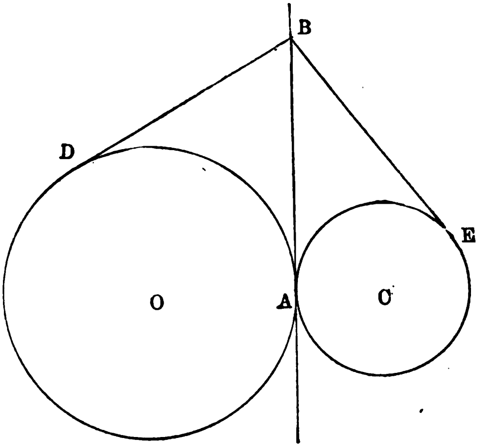
Corollary IV. *Of several circumferences tangent in the same point o the same straight line, the nearest to the straight line is that with the reatest radius.*

Scholium. A straight line is an arc of the circumference the radius f which is infinite.

Theorem 66.

The tangents drawn from a point of the intangent of two xternally tangent circumferences, are equal to each other.

Let O and C be two circumferences externally tangent in A; let A B be their intangent; let B D be tangent to the circumference O, and B E, a tangent to the circumference C.



Because B D and B A are tangent to the circumference O, the line B D is equal to B A [57]; because B E and B A are tangent to the circumference C, the line B E is equal to B A: therefore B D is equal to B E [i], which W. T. B. D. Inversely

The point from which are drawn unequal tangents of two externally tangent circumferences, does not belong to their intangent.

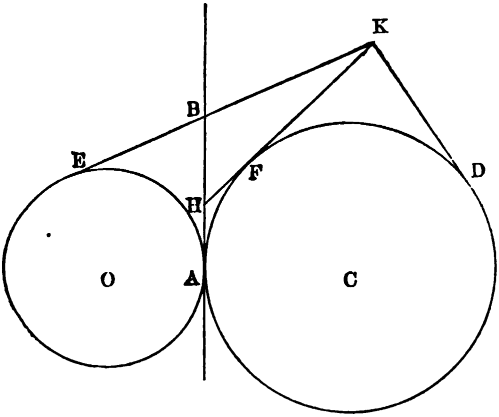
Corollary I. *The intangent of two externally tangent circumferences bisects their extangents.*

Corollary II. *The circumference described on an extangent to two externally tangent circumferences is tangent to their centre line in their point of tangence.*

Theorem 67.

The tangents drawn from a point without the intangent of two externally tangent circumferences are unequal.

Let the point K be without the intangent A B to the circumferences O and C externally tangent in A; let K D be a tangent to the circumference C, and K E, a tangent to the circumference O.



Let K H be a second tangent in F to the circumference C [54 i]; then B A will be greater than B H. Because K H is less than K B and B H together [6 i], much more is K F less than K B and B H together: now B E is equal to B A [57], and K D is equal to K F; therefore K D is less than K E, which W. T. B. D. Inversely

The point from which are drawn two equal tangents to two externally tangent circumferences, belongs to their intangent. (Conv. of 64.)

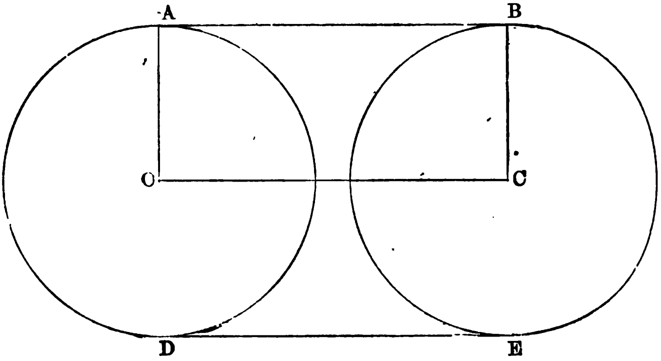
COROLLARY I. *If from a point on one side of their intangent, two tangents be drawn to two externally tangent circumferences, the tangent to the circumference on the same side of the common tangent is less than the other.*

COROLLARY II. *If from the same point there be two equal tangents to two externally tangent circumferences, the circumference drawn from that point as a centre with one of the tangents as a radius, is tangent to the centre line of the two first circumferences in their point of tangence.*

THEOREM 68.

Two extangents to equal circumferences are parallel to each other.

Let A B be a tangent to the circumferences O and C in A and B, and D E another tangent to the same circumferences in D and E.



Let the centre line O C be joined, and let the radii O A and C B join its extremities to A and B; then A O is equal to B C [37 ii]. Because A O and B C are perpendicular to A B [55], they are parallel to each other [16]: therefore A B is parallel to O C [21]. The same reasoning would prove that D E is also parallel to O C; therefore A B and D E are parallel to each other [17 iii], W. W. T. B. D. Inversely *Two circumferences are unequal if their extangents be not parallel to each other.*

COROLLARY I. Conversely, *two circumferences are equal to each other, if their extangents be parallel to each other.*

COROLLARY II. *If a tangent, or a secant, to one of two equal circumferences be parallel to the centre line, it is tangent, or secant, to the other.*

COROLLARY III. *The middle perpendicular on the centre line of two equal circumferences bisects their extangents and every common secant parallel to the centre line, or outer part of such secant.*

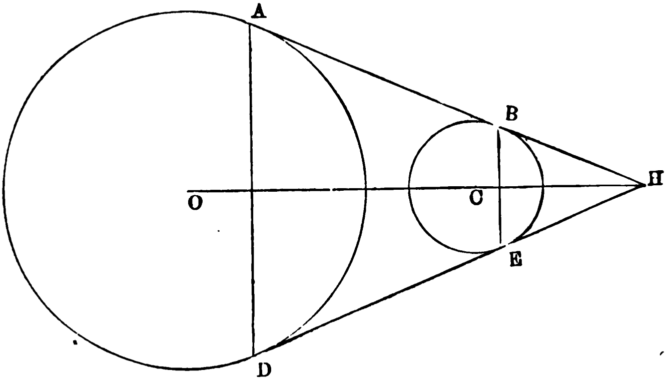
COROLLARY IV. *If two common secants to two circumferences be* *extangent to equal circumferences concentric to the first, they are equal to each other; also they shall be perpendicular to the chords joining their extremities, or those of their outer parts.*

COROLLARY V. *The common secant to two circumferences which passes through both centres is greater than any other.*

THEOREM 69.

Two extangents to the same circumferences are symmetric to the centre line.

Let A B and C D be two extangents to the circumferences O and C.



Let A D and B E be joined.

If A B be parallel to DE, then the circumferences O and C will be equal to each other [68 i]. Because A D and B E are diameters [55 ii] perpendicular [55] to the extangents, they are equal [37 ii] and parallel [16] to each other; therefore A B and D E are symmetric to the middle line of A D and BE [33 i], that is, to the centre line.

If A B and D E be not parallel to each other, then, if indefinitely produced, they will meet in a point H [18 i]. Let H O be joined. Because the tangents H A and H D are symmetric to O H [57 iv], and the tangents H B and H E are equal to each other [57], the points B and E are symmetric to O H. But the point C is equally distant from the points B and E; then H O passes through the point C [29 i]: therefore A B and D E are symmetric to O C, which W. T. B. D.

Corollary I. *There cannot be more than one extangent to two circumferences on each side of the centre line.*

Corollary II. *If the extangents of two circumferences be not parallel to each other, they shall, if indefinitely produced, cut each other on the continuation of the centre line.*

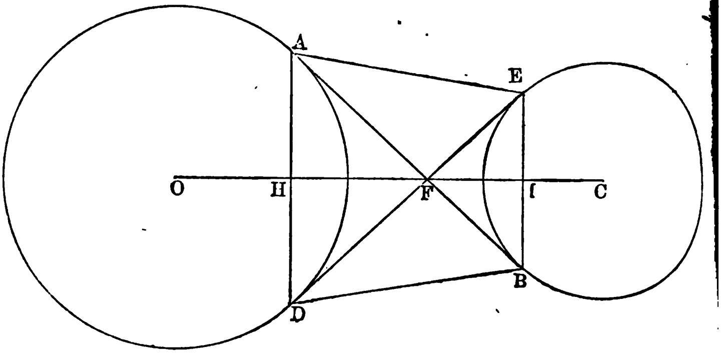
Corollary III. *The tangent to one of any two circumferences, through a point of extangence, is tangent to the other.*

Corollary IV. *If two common secants to two circumferences be extangents to two circumferences concentric to the first, they are symmetric to the centre line.*

Theorem 70.

Two intangents to the same circumferences are symmetric to the centre line.

Let A B and D E be two intangents to the circumferences O and C, and let F be their intersection.



Let A D and B E be joined, and let H I be their middle line; let also A E and DB be joined. Because F A is equal to F D and F B to F E [57], the lines A E and D B are symmetric to H I [35]. Because H I is the ldle perpendicular on A D and E B [28], its continuation will pass ough the centres O and C [44 i.]; thence A E and D B are sym-tric to the centre line O C; therefore A B and D E are symmetric O C [32 i] which W. T. B. D.

Corollary I. *Neither of two circumferences has more than one point intangence on each side of the centre line.*

Corollary II. *The tangent to one of any two circumferences through point of intangence, is tangent to the other.*

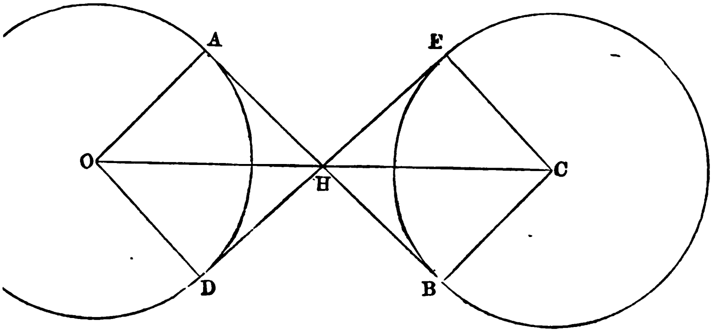
Corollary III. *If from the intersection of an intangent to two cir-umferences with their centre line a tangent be drawn to one of them, its ntinuation is tangent to the other.*

Corollary IV. *The common secants through the points of intan-nce of two circumferences, or their outer parts, are symmetric to the ntre line.*

Corollary V. *If two common secants to two circumferences be in-ngent to circumferences concentric to the first, they are symmetric the centre line.*

Theorem 71.

The intangents of two equal circumferences bisect each other the centre line.



et A B and D E be two intangents to the circumferences O . C.

Let the centre line O C be joined, and let the centres be joined to the points of intangence: then, because A B and D E are symmetric to the centre line [70], they will cut each other in a point H thereof [30i]. Because the circumferences O and C are equal to each other [*Hyp.*], A O is equal to B C [37 ii]; because A O and B C are both perpendicular to A B [55], they are parallel to each other [16]: therefore O C and A B bisect each other [25 i]; that is, H is the middle point of A B. The same reasoning would prove that H is also the middle point of D E; therefore A B and D E bisect each other on the centre line, W. W. T. B. D. Inversely

Two circumferences the intangents of which cut each other in unequal parts, are not equal to each other.

COROLLARY I. Conversely, *two circumferences the intangents of which bisect each other, are equal to each other.*

COROLLARY II. *The intangents of two equal circumferences bisect the centre line.*

COROLLARY III. *The common secants through the points of intangence of two equal circumferences are parallel to each other; also they are perpendicular to the chords joining their extremities, or those of their outer parts*, and conversely.

COROLLARY IV. *If from the middle point of the centre line of two equal circumferences, a tangent be drawn to one of them, its prolongation is tangent to the other.*

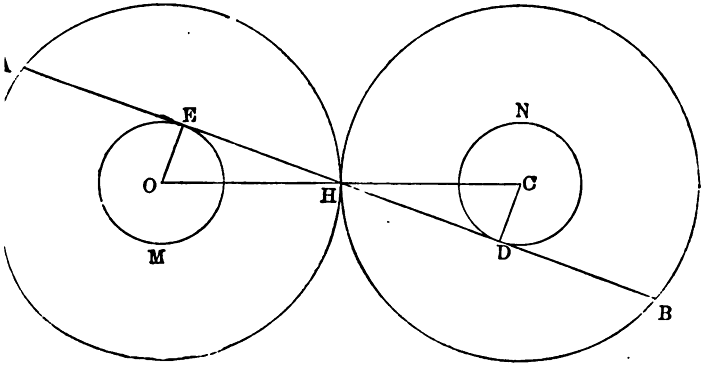
Scholium. Each intangent of two equal circumferences is bisected by the centre line.

THEOREM 72.

The intercepts of a common secant through the points of tangence of two equal tangent circumferences, are equal to each other.

Let two equal circumferences O A H and C B H be tangent in H, and let A B be a common secant through the point H.

Let O E and C D be two perpendiculars to A B from the centres O and C, and let O E M and C D N be two circumferences drawn conntric to the first with O E and C D as their respective radii; let so the centre line O C be joined: then A B is intangent to the rcumferences O E M and C D N.



Because O H is equal to C H [37 ii], the two circumferences E M and C D N are equal to each other [71 ii]; therefore O E is qual to C D. The chords A H and H B are then equally distant om the centres O and C [11]: therefore they are equal to each her [49 i], W. W. T. B. D. Inversely

If the intercepts of a common secant through the point of tangence of o tangent circumferences be unequal, the circumferences are not ual to each other.

Corollary I. *If the intercepts of a common secant through the point tangence of two tangent circumferences be equal to each other, the cumferences are equal to each other.*

Corollary II. *The intercepts of a common secant through the middle int of the centre line of two equal circumferences are equal to each er,* and conversely.

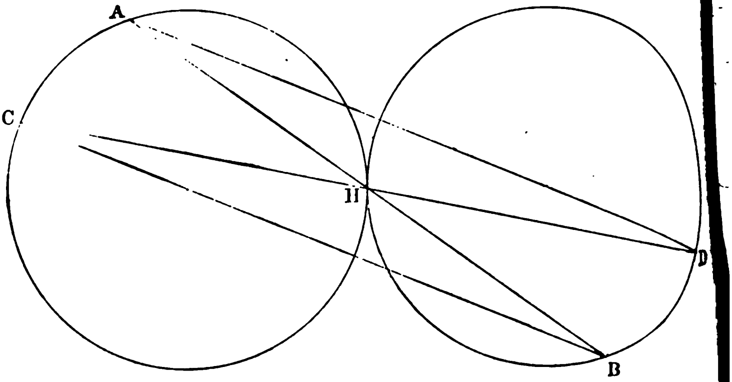
Corollary III. *The centre line of two equal circumferences bisects common secant passing through its middle point,* and conversely.

Corollary IV. *The continuation of the secant to one of two equal cumferences through the middle point of the centre line, is secant to other.*

THEOREM 73.

The common secants which join the extremities of two common secants passing through the point of tangence of two equal tangent circumferences, are parallel and equal to each other.

Let A C H and B D H be two equal circumferences tangent in H; let A B and C D be two secants through the point H, and let A D and C B be joined.



Because the straight lines A B and C D are both bisected in H [72], they bisect each other; therefore A D and C B are parallel [26] and equal [26 i] to each other, W. W. T. B. D.

COROLLARY I. *The chords which join the extremities of two common secants passing through the point of tangence of two equal tangent circumferences, are parallel and equal to each other.*

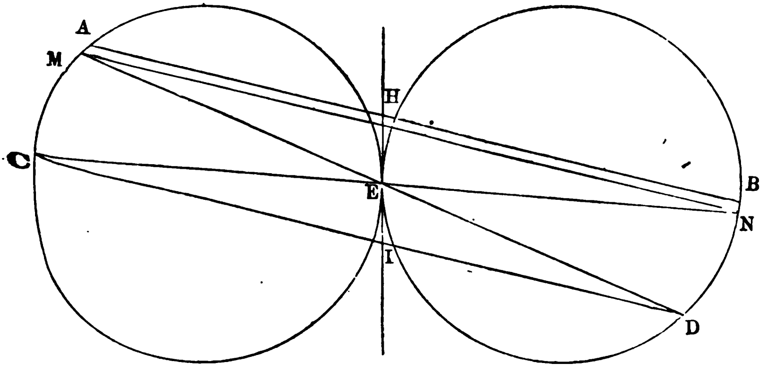
COROLLARY II. *The common secants, or the chords, which join the extremities of two common secants passing through the middle point of the centre line of two equal circumferences, or the extremities of their outer parts, are parallel and equal to each other.*

COROLLARY III. *The common secants which join the extremities of two common secants passing through the point of tangence of two equal circumferences cut the intangent at equal distances from the point of tangence.*

THEOREM 74.

If two parallel common secants to two equal tangent circumferences be equally distant from the point of tangence, the common secants joining their extremities cut each other in the point of tangence.

Let A C E and B D E be two equal circumferences tangent in E, and let A B and C D be two parallel common secants equally distant from the point E.



Let H I be the intangent of the two circumferences, cutting A B in H, and C D in I: then H E is equal to E I [23]. Let C E and D E be joined and produced to N and M, and let M N be joined. Because M N is parallel to C D [73] and passes through the point H [73 iii], it coincides with A B [17 iv]; thence the points M and N coincide respectively with the points A and B: that is, the continuation of D E passes through the point A, and that of C E, through the point B; therefore the straight lines joining A to D and B to C cut each other in E, which W. T. B. D.

COROLLARY I. *If two parallel common secants to two equal tangent circumferences be equally distant from the point of tangence, they are equal to each other.*

COROLLARY II. *The parallel common secants through the points of section of two equal secant circumferences, are equal to each other.*

PROBLEMS.

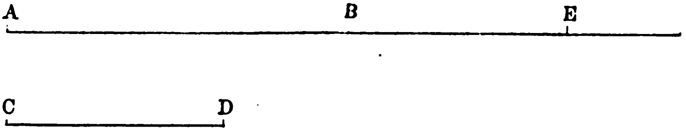
The following problems are here solved as examples for students.

The only instruments allowed for the solution of problems are the *Ruler* and *Compasses;* the Ruler being used for drawing and producing straight lines, the Compasses for describing circumferences and transferring distances.

Problem 1.

To find the sum of two given straight lines.

Let A B and C D be the two given straight lines.



Produce A B indefinitely and take, on the prolongation, B E equal to C D; the line A E is the required sum.

Because B E is equal to C D, the sum of A B and B E is equal to that of A B and C D [ii]; thence A E is equal to the sum of A B and C D; therefore the sum of the given straight lines A B and C D has been found, which was to be done.

Problem 2. *To find the sum of several straight lines.*

Problem 3. *To find the product of a given straight line by a given number.*

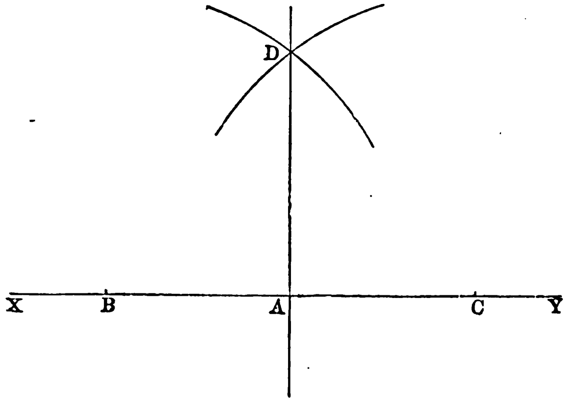
Problem 4. *To find the difference of two given straight lines.*

Problem 5.

Through a point given in a given straight line, to draw a perpendicular to that line.

Let X Y be the given straight line and A the given point.

Take the points B and C on X Y at equal distance from the point ; from the centres B and C, with a radius greater than half of B C,



describe two circumferences: they will cut each other in a point D [62]. Draw a straight line through the points D and A: the line A D is the required perpendicular.

Because the point D belongs to both the equal circumferences B and C, its distance from the centres B and C is equal to the common radius; therefore the point D is equally distant from the points B and C. Because AB and A C are equal to each other [*Const.*], the line A D has the points A and D, equally distant from the points B and C; thence A D is the middle perpendicular to A B [14]: that is, A D is a perpendicular to X Y: therefore through the point A given in the given straight line X Y, a perpendicular has been drawn, W. W. T. B. D.

PROBLEM 6. *To draw the middle perpendicular to a given straight line.*

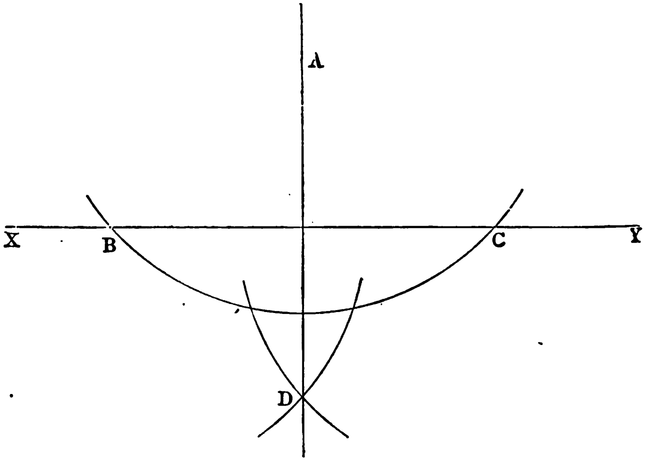
PROBLEM 7. *To bisect a given straight line.*

PROBLEM 8. *To bisect a given arc.*

PROBLEM 9.

Through a point given without a given straight line, to draw a perpendicular to that line.

Let X Y be the given straight line and A the given point.



From the centre A, with a radius greater than the distance of A to X Y, describe a circumference cutting X Y in two points B and C; from the centres B and C, with a radius greater than half of B C describe two circumferences cutting each other in D, and through the points A and D draw a straight line: the line A D is the required perpendicular.

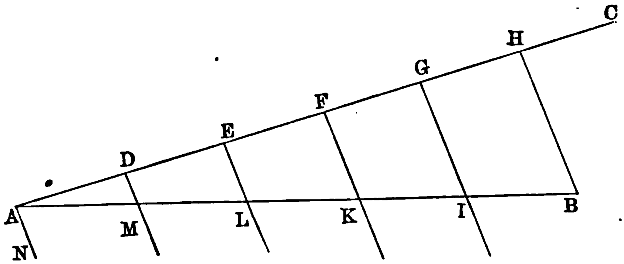
Because B and C are points of the same circumference, their distances from the centre A are equal to each other; therefore the point A is equally distant from B and C. Because the point D belongs to both the equal circumferences B and C, its distances from the centres B and C are equal to each other; thence the line A D has two points A and D equally distant from the points B and C, and the middle perpendicular to B C [14]: therefore A D is perpencular to X Y,[1] which W. T. B. D.

PROBLEM 10. *Through a point given without a given straight line, to raw a parallel to that line.*

PROBLEM 11.

To divide a given straight line into a given number of equal rts.

Let A B be the given straight line to be divided into *five* equal rts.



Through the point A, draw the straight line A C in any direction; ce any distance A D and set it off upon A C five times successively A D, D E, E F, F G and G H; join H B, and through the points F, E and D draw G I, F K, E L and D M parallel to H B [*Probl.*]: the line A B is divided by the points M, L, K and I into five al parts.

Let A N be a parallel to H B. Because the intercepts of A C ween parallels H B, G I, F K, E L, D M, and A N, are equal each other [*Const.*], they are equally distant from each other ii]; thence the intercepts of A B between the same parallels equal to each other [23 i]: therefore the straight line A B is divided by the points M, L, K and I into five equal parts, W. W. T. B. D.

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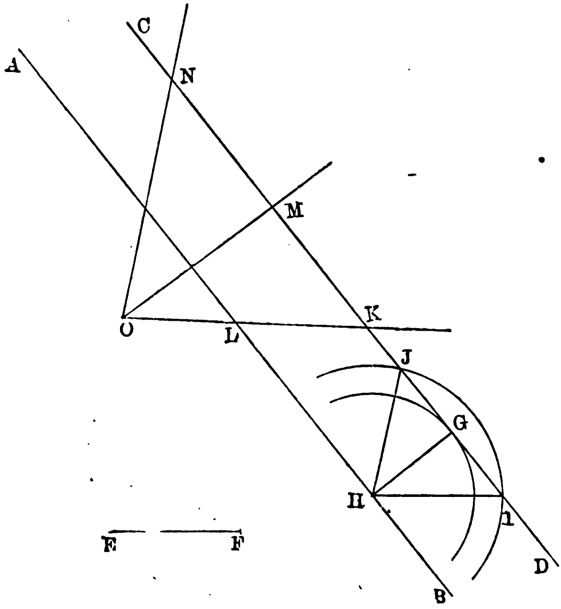
[1] As in the theorems, the student must supply here, and in the demonstration of most r problems, the rest of the reasoning suppressed for the sake of brevity. Also the ent should find, for each problem, other solutions than those given in The Elements.

PROBLEM 12.

(*Discussed.*)

Through a given point, to draw a transversal to two given parallels between which the intercept shall be equal to a given straight line.

Let O be the given point, A B and C D the given parallels, and E F the straight line to which the intercept of the required transversal must be equal.



From any point H of A B, with a radius equal to E F, describe a ircumference cutting C D in a point I; join H I, and through the oint O draw O K parallel to H I [*Probl.* 10]: the line O K is the equired transversal.

Let L K be the intercept of O K between the parallels A B and D. Because O K and H I are parallel to each other [*Const.*], the ntercepts L K and H I between the parallels A B and C D are equal o each other [22]. But H I is equal to E F [*Const.*]; then L K s equal to E F [i]: therefore O K is a transversal, the intercept L K of which, between the parallels A B and C D, is equal to E F, which W. T. B. D.

The given straight line E F may be equal to the distance of the parallels A B and C D from each other, or it may be greater, or smaller.

Let H G and O M be perpendicular to C D: then, they are also perpendicular to A B [17]. If E F is equal to the distance of the parallels A B and C D from each other, it is equal to H G [11,] then he circumference described from the point H with a radius equal to E F will be tangent to C D in the point G [52]: but the intercept of M between A B and C D is equal to H G [19]; therefore O M will e the only transversal satisfying the data.

If E F is greater than the distance of A B and C D from each other t is greater than H G [11], and the circumference described from the oint H, with a radius equal to E F, will cut C D in two points I and ; therefore, if H I and H J be joined, and parallels O K and O N to hem be drawn through the point O, there will be two transversals, K and O N, satisfying the data of the problem.

If E F is smaller than the distance of A B and C D from each her, it is smaller than H O [11], then the circumference described om the point H with a radius equal to E F will not touch the line D: therefore there will be no transversal satisfying the data of the oblem; that is, the problem will be impossible.

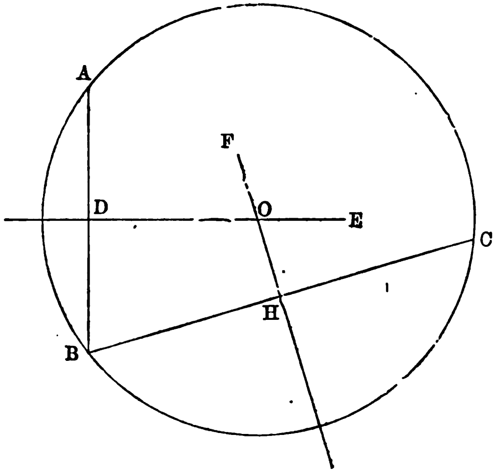
Therefore the problem may have two solutions, or one only, or ne.

PROBLEM 13.

(*Eucl.* III. 1.)

Through three given points, to describe a circumference.

Let A, B and C be the three given points.



Join A B and B C; draw D E middle perpendicular to A B [*Probl.* 6], and H F middle perpendicular to B C; from the intersection O of D E and H F, describe a circumference through the point A: the circumference O is the required circumference.

Because the point O belongs to the middle perpendicular to A B, it is equally distant from the points A and B [14]; therefore the circumference O will pass through the points A and B. Because the point O belongs to the middle perpendicular to B C, it is equally distant from the points B and C; thence the circumference O will pass through the points B and C: therefore the circumference O will pass through the points A, B and C, which W. T. B.D.

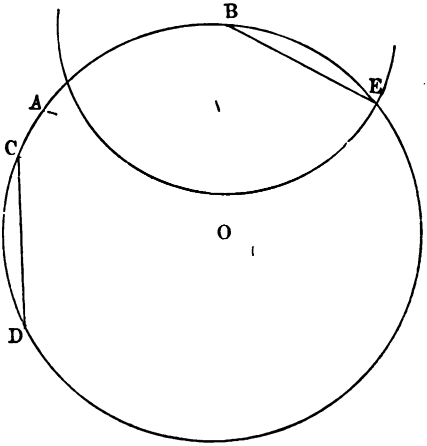
PROBLEM 14. *To find the centre of a circumference, or an arc.*

PROBLEM 15. *To complete the circumference of a given arc.*

PROBLEM 16.

To find the sum of two arcs given in a given circumference.

Let O be the given circumference and A B, C D, the two given arcs.



Join C D, and from the centre B, with a radius equal to C D describe a circumference cutting the circumference O in E: the arc A E is the required sum.

Join B E. Because B E is a radius of the circumference B, the line B E is equal to C D [*Const.*]; because the arc B E is equal to the arc C D [43 i], the sum of the arcs A B and B E is equal to that of the arcs A B and C D [ii]; therefore the arc A E is equal to the sum of the arcs A B and C D, which W. T. B. D.

PROBLEM 17. *To find the sum of two arcs given in two given equal circumferences.*

PROBLEM 18. *To find the difference of two arcs given in a given circumference, or in two given equal circumferences.*

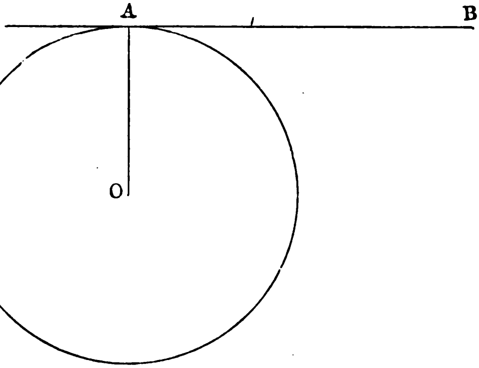
PROBLEM 19. *To find the product of a given arc by a given number.*

Problem 20.

(*Eucl.* III. 17.)

Through a point given in a given circumference, to draw a tangent to the circumference.

Let O be the given circumference and A the given point.



Join the point A to the centre O, and draw A B perpendicular to A O [*Probl.* 5]: the line A B is the required tangent.

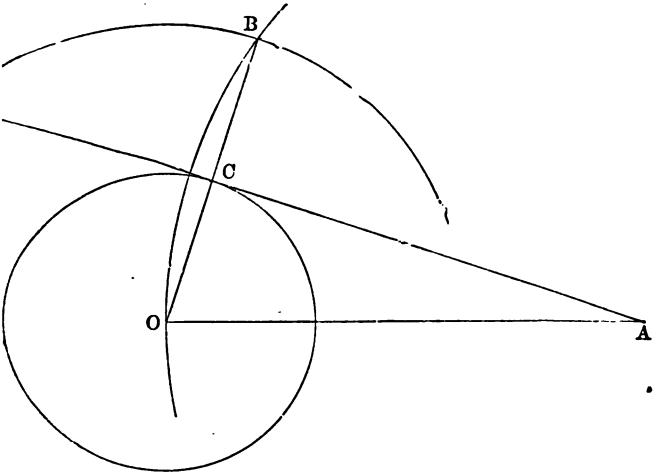
For A B is perpendicular at the extremity of the radius O A [*Const.*]; therefore it is tangent to the circumference O in the point A [53], which W. T. B. D.

Problem 21.

Through a point given without a given circumference, to draw a tangent to the circumference.

Let O be the given circumference and A the given point.

Join A O; from the centre A, with a radius equal to A O, describe a circumference A. From the centre O, with a radius double of that of the circumference O, describe another circumference cutting the circumference A in a point B. Join O B, cutting the circumference ı a point C, and draw a straight line through the points A and C; line A C is the required tangent.



Because the radius O C is half of O B [*Const.*], and O B is a chord the circumference A, the line A C is perpendicular to O B [44 i]; at is, A C is perpendicular at the extremity of the radius O C: erefore A C is a tangent to the circumference O [53], which ∵T. B. D.

BOOK II.

ON ANGLES AND PROPORTIONAL LINES.

BOOK II.

N ANGLES AND PROPORTIONAL LINES.

CHAPTER I.

ON ANGLES.

SECTION I.

ON ANGLES GENERALLY.

DEFINITIONS.

> **"A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction." EUCLID.**

> **"When two straight lines meet, the greater or lesser quantity by which they recede from each other, as to their position, is called an angle." LEGENDRE.**

> **"An angle is the infinite portion of a plane contained between two straight lines which cut each other." BERTRAND (of Geneva).**

> **"The figure formed by two straight lines which cut each other is called an angle." BLANCHET.**

1. An *angle* is the figure formed by two straight lines hich have one extremity in common; the straight lines hich form the angle are its *sides*, and their common ex-emity is the *vertex* of the angle, or the *angular* point.

As the straight line is infinite in magnitude and only one extremity of the es of an angle is determined, the other extremity is at an infinite distance m the vertex.

An angle is distinguished by the letter at its vertex. If two or more angles have the same vertex, each angle is distinguished by a letter at some point of each side, together with the letter at the vertex, taking care to read the latter between the two others.

2. The *angular* magnitude is the difference in the direction of the sides of the angle.

Thus the magnitude of an angle is wholly independent of the length of its sides, but increases or diminishes according to their directions; so that the sides of an angle may be shortened or lengthened, made equal or unequal, without altering the magnitude of the angle itself.

From the preceding definitions, it follows, that two angles are equal when the difference in the directions of their sides is the same. It is obvious that two angles, the sides of which coincide, are equal in magnitude, since the difference in direction of their sides is one and the same; also, if an angle is applied upon its equal, so that the vertex and one of the sides coincide, the other sides coincide also, and then the two figures form only one angle, although the other extremities of the sides may not coincide.

If two indefinite straight lines have one extremity in common, one of them may be conceived to turn on their common point, and have at first a direction very little different from that of the other, so that the angle formed by these lines is very little, if reckoned from one side of one of the two lines, but is much greater if reckoned from the other side of the same line. Thus every angle has two magnitudes, one of which increases when the other diminishes, and diminishes when the other increases.[1]

3. Each of the two magnitudes of an angle is the *explement* of the other. Two angles are *explementary* when the magnitude of one is equal to the explement of the other; each of the two angles is also the *explement* of the other.

An angle contained in a plane divides it into two portions, one of which increases or diminishes with the angle, whilst the other increases or diminishes with the explement of the angle.

4. The portion of the plane containing an angle, which

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[1] Throughout The Elements, when the magnitude of an angle is spoken of without anything further being said, the *lesser* magnitude only is meant.

ncreases or diminishes as the angle increases or diminishes, s the *angular space*. All points of the angular space are *nside* the angle, or in the *opening* of the angle; all points f the plane not contained in the angular space, are *outside* he angle.

The two sides of an angle may be oblique or perpendicular to one another; hey may even be the continuation of each other; that is, have opposite irections, so that they form a straight line: in the latter case the angle is alled a *straight* angle. Thus

5. A *straight* angle is an angle each side of which is the ontinuation of the other. An angle which is less than a traight angle is a *projecting* angle; an angle which is reater than a straight angle is a *receding* angle.

Hence, the two sides of a straight angle form a straight line.

6. Two angles which have the same vertex and one common side are *adjacent to each other*, or simply *adjacent*.

If one side of an angle be produced beyond the vertex, the two angles thus rmed, having one side in common, will be adjacent to each other. If both ides of an angle be produced beyond the vertex, each of the four angles us formed with a common vertex will be adjacent to two others; the oppoite angles, that is those that are not adjacent, are said to be the *vertical* of ach other. Thus

7. *Vertical* angles are two angles such that the sides f the one are the continuation of the sides of the other eyond the vertex.

8. The line which joins two points of the sides of an ngle *subtends* the angle.

9. Two points of the sides of an angle, which are equally istant from the vertex, are *symmetric* points of the angle.

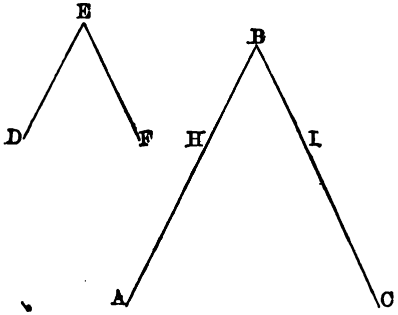
The straight line which joins two symmetric points of an angle subtends the angle *symmetrically.*

10. The *bisectrix* of an angle is the straight line which divides the angle into two equal parts.

It follows from the definition of the angle that the bisectrix passes through the vertex.

THEOREM 1.

Similar angles are equal to each other.



Let there be two similar angles A B C and D E F.

If the sides A B and B C be equal to the sides D E and E F respectively, let the two angles be applied to each other; then, because the two angles have the same form, they will coincide with each other: therefore they are equal to each other.

If the sides of the angle A B C be not equal to those of the angle D E F, let B H be equal to E D and let B I be equal to E F; then the angles H B I and D E F are equal to each other [*Dem.*]; therefore the angles A B C and D E F are equal to each other, which was to be demonstrated. Inversely

Unequal angles are not similar to each other.

THEOREM 2.

Equal angles have equal explements.

Because they are equal to each other, they may be made to coincide; their explements will then be one and the same for all of them. Inversely

Angles which have unequal explements are unequal.

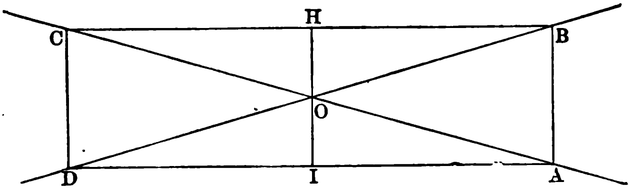
Corollary. Conversely, *angles which have equal explements are equal to each other.*

Theorem 3.

Eucl. I. 15.)

Vertical angles are equal to each other.

Let A O B and C O D be two vertical angles.



Then A O C and B O D are two straight lines [*Def.* 7]. Let O A, O B O C, and O D be equal to one another; let A B, C D, B C, and A D be joined; and let H I be the middle line of B C and A D; then H I passes through the point O [I. 26 ii]. Because A B and D C are symmetric to H I [I. 33 iii], the lines A O and O D are symmetric to H I; also, B O and O D are symmetric to the same line. Now, let the plane containing the figure be folded upon H I; then O A will coincide with its symmetric O D, and O B with O C: that is, the angle A O B will coincide with the angle C O D; therefore the angles A O B and C O D are equal to each other, W. W. T. B. D. Inversely

If two angles having the same vertex be unequal, and one side of one of them be the continuation of one side of the other, the other sides are not the continuation of each other.

Corollary I. *If one of the angles formed by two straight lines cut-*

ting each other be equal to its adjacent, the four angles are equal to one another.

Corollary II. *The bisectrix of an angle bisects its vertical and its explement.*

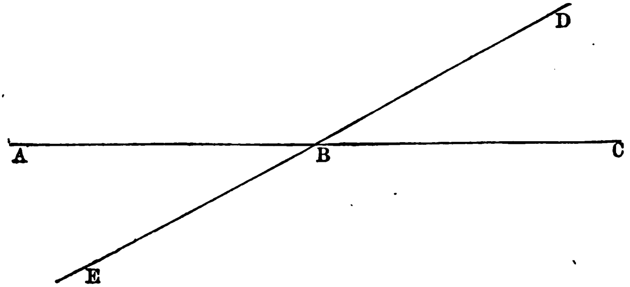
Corollary III. *The prolongations of the sides of a projecting angle are outside the angle,* and conversely.

Corollary IV. *The prolongations of the sides of a receding angle are within the angle,* and conversely.

Theorem 4.

A straight angle is equal to its explement.

Let A B C be a straight angle.



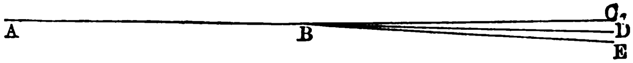
Then the straight line B C is the continuation of the straight line A B; that is, A C is a straight line. Let the angle A B C be divided into two parts by the straight line B D, and let B D be produced to E beyond the vertex B; then the angle A B E is equal to its vertical D B C [3], and the angle E B C is equal to its vertical A B D: therefore, the angles A B E and E B C are together equal to the angles A B D and D B C [iv]; that is, the straight angle A B C is equal to its explement, W. W. T. B. D.

Corollary. *The explement of a projecting angle is a receding angle,* and conversely.

THEOREM 5.

Conversely, *an angle equal to its explement is straight.*

Let A B C be an angle equal to its explement.



Let B D be the prolongation of A B beyond the vertex B; then BD is a straight line. Let the plane containing the angle be lded upon A B D; then all points of A B D will remain in their st position [I. 2] and B C will take a position B E on the other de of B D; therefore the angle A B C will be equal to the angle B E. But the angle A B C is equal to its explement [*Hyp.*], that , to A B E and E B C together: then the angle A B E is also equal the angles A B E and E B C together [i]; therefore the angle B C has no magnitude at all and the lines B C and B E coincide ith B D. But A B D is a straight line [*Const.*]; then A B C is a raight line, or, B C is the continuation of A B: therefore the angle B C is straight, W. W. T. B. D.

COROLLARY. *If the opposites of four angles having the same vertex be ual to each other, they are vertical.*

Scholium. The bisectrices of vertical angles are the continuation each other (2 ii).

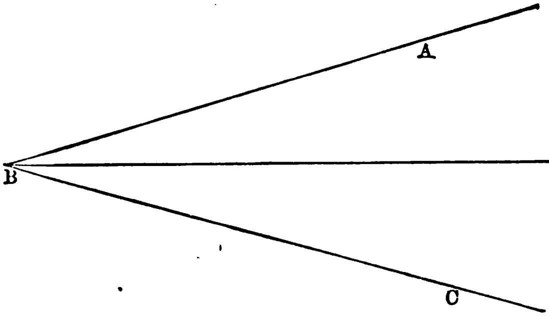
THEOREM 6.

Two points of the sides of an angle equally distant from the rtex are symmetric to the bisectrix of the angle.

Let the points A and C of the angle A B C, be equally distant from e vertex B.

Let B D be the bisectrix of the angle A B C, and let the plane ntaining the angle be folded upon B D. Because the angles

A B D and D B C are equal to each other [*Hyp.*], the side A B will coincide with the side C B, and because B A is equal to B C, the point A will coincide with the point C; therefore the points A and C are symmetric to B D [I. *Def.* 22], which W. T. B. D. Inversely



Two points of the sides of an angle which are not symmetric to the bisectrix, are unequally distant from the vertex.

Corollary I. Conversely, *two points of the sides of an angle which are symmetric to the bisectrix are symmetric points of the angle.*

Corollary II. *The portions of the sides of an angle contained between the vertex and symmetric points, or between symmetric points are symmetric to the bisectrix.*

Corollary III. *The bisectrix of an angle bisects every angle formed by lines joining any point thereof, to symmetric points of the angle.*

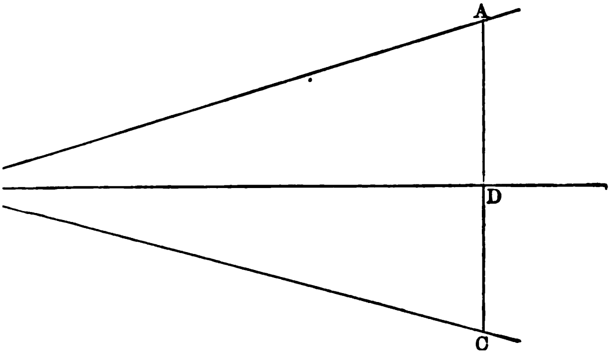
Corollary IV. *The lines joining the symmetric points of vertical angles, are symmetric to the common bisectrix of the angles.*

Theorem 7.

The bisectrix of an angle is the middle perpendicular on every straight line subtending the angle symmetrically.

Let A C be a straight line subtending symmetrically the angle A B C, and let B D be its bisectrix cutting A C in D.

ints A and C, being symmetric points of the angle A B C are symmetric to the bisectrix B D [6]; then B D is the erpendicular on A C [I. 28] which W. T. B. D.



LARY I. Conversely, *the middle perpendicular on the straight ending an angle symmetrically, is the bisectrix of the angle.*

LARY II. *The perpendicular drawn from the vertex on the line subtending an angle symmetrically, bisects the latter and angle.*

LARY III. *The straight line perpendicular to the bisectrix of an ts the sides in symmetric points.*

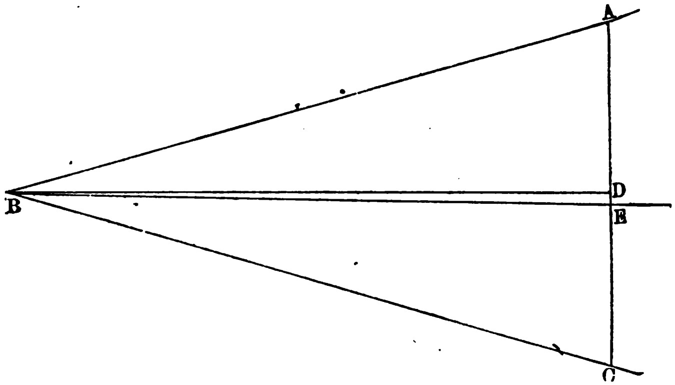
LARY IV. *Two straight lines subtending an angle symmetric- parallel to each other.*

LARY V. *If two straight lines subtend an angle symmetrically, joining their unlike extremities cut each other on the bisectrix.*

THEOREM 8.

line joining the vertex to the middle point of a straight tending an angle symmetrically, is the bisectrix of the

and C be two symmetric points of an angle A B C, joined by the straight line A C, and B D a straight line joining the vertex to the middle point of A C.



Then A D is half of the straight line A C. Let B E be the bisectrix of the angle A B C, cutting A C at the point E; then E is the middle point of A C [7], and A E is half of A C. But A D is already equal to half of A C; therefore A D is equal to A E [i], and the points D and E are one and the same point. Because the straight lines B D and B E, have two points in common, they coincide throughout with each other; therefore B D bisects the angle A B C, which W. T. B. D.

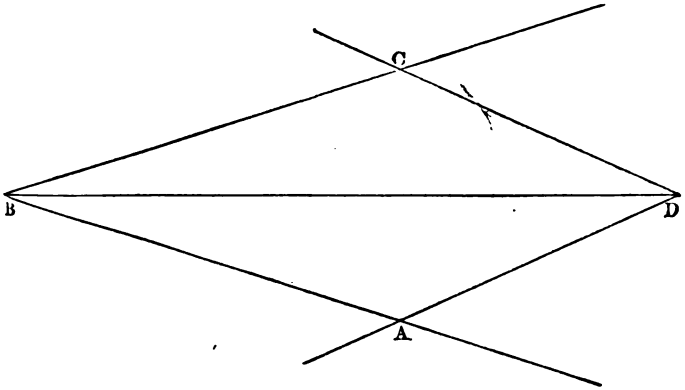
Corollary. *The straight line passing through the middle points of two straight lines subtending an angle symmetrically, passes also through the vertex and bisects the angle.*

Theorem 9.

If the sides of two angles having the same bisectrix, cut each other, the points of section are symmetric points of both angles.

Let B D be the bisectrix of the two angles A B C and A D C, the sides of which cut each other in A and C.

Let the figure be folded upon B D; the angle A B D being equal the angle C B D, the line B A will coincide with B C; also D A



will coincide with D C, and they will cut the coinciding lines B C and B A at the same point: that is, the points A and C will coincide with each other; thence B A is equal to B C and D A is equal to D C; therefore the points A and C are symmetric points of both angles, W. W. T. B. D.

Corollary I. Conversely, *if the sides of two angles cut each other in symmetric points, the line passing through the vertices is the bisectrix of both angles.*

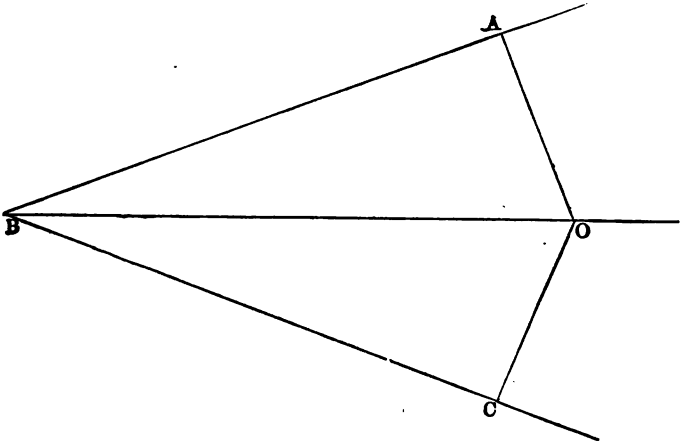
Corollary II. *If the sides of two angles, having the same bisectrix, cut each other, the points of section are symmetric to the bisectrix,* and conversely.

Theorem 10.

Each point of the bisectrix of an angle is equally distant from the sides of the angle.

Let A B C be an angle and B O its bisectrix.

From any point O of the bisectrix B O let there be drawn O A and O C, perpendicular on B A and B C respectively, and let the figure be folded upon B O. Because the angles O B A and O B C are equal to each other [*Hyp.*], the lines B A and B C will coincide with each other, and also the perpendiculars O A and O C [I. 10]; the latter are then equal to each other [I. 3 i]: but the perpendiculars O A and O C measure the distance from the point O to the lines B A and B C [I. 11]; therefore the point O is equally distant from the sides of the angle, W. W. T. B. D. Inversely



Any point not equally distant from the sides of an angle does not belong to its bisectrix.

Corollary I. *Each point of the bisectrix of an angle is equally distant from symmetric points of the angle.*

Corollary II. *The projections of a point of the bisectrix, on the sides of an angle, are symmetric points of the angle,* and conversely.

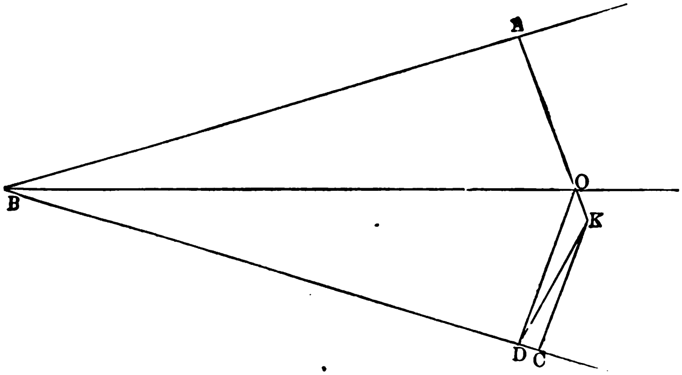
Corollary III. *The intersection of the bisectrices of two angles formed by a straight line with two others, is equally distant from the three lines.*

Scholium. Each point of the bisectrix of a straight angle is equally distant from symmetric points of the angle (I. 14).

THEOREM 11.

Any point without the bisectrix of an angle is unequally distant from the sides of the angle.

Let B O be the bisectrix of an angle A B C, and K any point without it.



Let K A and K C be perpendicular on B A and B C; let O be the point O where K A cut the bisectrix; let also O D be a perpendicular on B C, and let K D be joined: then K D is less than K O and O D together [I. 6 i]. Because O A and O D are equal to each other [10], K D is less than K O and O A together, that is, less than K A: but K C is less than K D [I. 11]; much more is K C less than K A; therefore the point K is unequally distant from the sides of the angle A B C, which W. T. B. D. Inversely

Every point equally distant from the sides of an angle belongs to the bisectrix of the angle.

COROLLARY I. *Every point equally distant from symmetric points of an angle belongs to the bisectrix of the angle.*

COROLLARY II. *The straight line which has two of its points equally distant from the sides, or from symmetric points, of an angle, is the bisectrix of the angle.*

COROLLARY III. *The straight line passing through the vertex, and one* *point of which is equally distant from the sides, or from symmetric points, of an angle, is the bisectrix of the angle.*

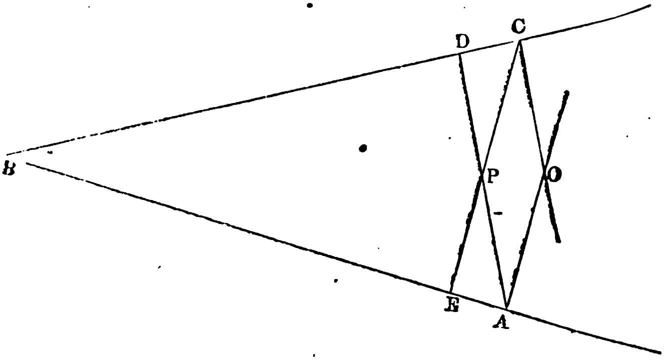
COROLLARY IV. *Every point of a line which divides a projecting angle into unequal portions, is further distant from the side with which it forms the greater angle,* and conversely.

Scholium. Every point equally distant from symmetric points of a straight angle belongs to the bisectrix of the angle (I. 15).

THEOREM 12.

Symmetric points of an angle are each equally distant from the other side of the angle.

Let A and C be symmetric points of the angle A B C.



Let A D be perpendicular on B C, and C E perpendicular on A B, the point P being their intersection; also let C O be perpendicular on B C, and A O perpendicular on B A, the point O being their meeting point. Because the lines A O and C O are perpendicular on the sides of the angle A B C, through symmetric points, their meeting point O is a point of the bisectrix of the angle [10 ii]; thence O A is equal to O C [10]. The lines O C and A D, being both perpendicular to B C, are parallel to each other [I. 16]; also O A and C E are parallel to each other; then A P is equal to O C [I. 22], and C P is equal to O A; therefore A P is equal to C P [i].

The point P, being equally distant from symmetric points of the gle A B C, belongs also to the bisectrix of the angle [11 i]; there- re P D is equal to P E [10]. But A P is equal to C P [*Dem.*]; then P and P D are together equal to C P and P E [iv]: that is, AD is ual to C E; therefore the distance from the point A to the side C is equal to the distance from the point C to the side B A [I. 11], hich W. T. B. D. Inversely

Points unequally distant from the sides of an angle are not symmetric ints of the angle.

COROLLARY. *The projections of each of symmetric points of an angle the other side are symmetric of the angle.*

SECTION II.

ON RIGHT ANGLES.

DEFINITIONS.

11. A *right* angle is one of which the sides are perpen- icular to one another. An *obtuse* angle is that which is reater than a right angle; an *acute* angle is that which is ss than a right angle.

12. When two angles are together equal to a right angle ey are *complementary* angles, and each of them is the *com- ement* of the other; when two angles are together equal to o right angles, they are *supplementary* angles, and each of em is the *supplement* of the other.

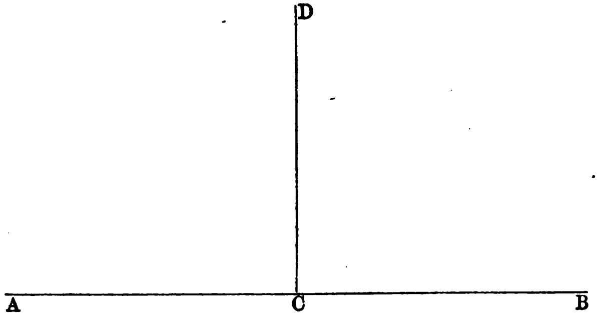
THEOREM 13.

(*Eucl. I. Def.* 10.)

If a straight line form with another two adjacent angles ual to each other, it is perpendicular to it.

Let C D be a straight line forming with the straight line A B two jacent angles, A C D and D C B, equal to each other.

Then C D is the bisectrix of the straight angle A C B, formed by the two portions C A and C B of the line A B. Let A and B be



symmetric points of the straight angle A C B; then any point D of the bisectrix C D is equally distant from A and B [10 i]. Now the straight line C D has two of its points, C and D, equally distant from the extremities of A B; therefore C D is perpendicular to A B [I. 15 ii], which W. T. B. D. Inversely

An oblique to a straight line forms with the latter unequal adjacent angles.

COROLLARY. *If two straight lines cut each other, the bisectrices of two adjacent angles are perpendicular to each other.*

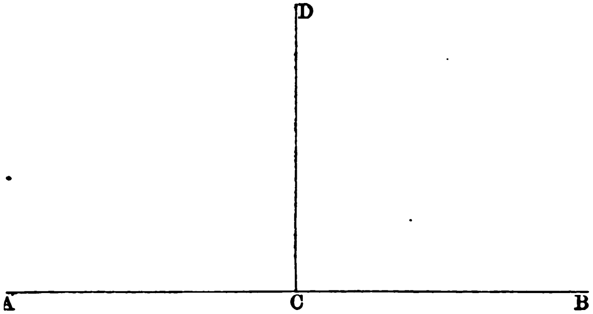
THEOREM 14.

Conversely, *a perpendicular to a straight line forms with the latter two adjacent angles equal to each other.*

Let C D be a perpendicular on the straight line A B, and let two points A and B of A B be equally distant from the point C.

The straight line C D being the middle perpendicular on A B, any point D of it is equally distant from A and B [I. 14]. But A and B are symmetric points of the straight angle A C B formed by the two portions C A and C B of the line A B; then C D, passing through the vertex C, and having the point D equally distant from the symmetric points A and B, is the bisectrix of the angle A C B [11 iii]:

re the angles A C D and D C B are equal to each other, T. B. D. Inversely



straight line form with another two adjacent angles not equal other, it is oblique to the latter.

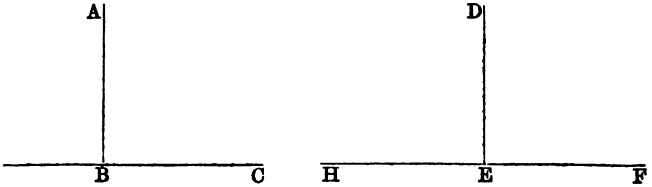
LLARY. *If two straight lines cut each other perpendicularly, the ngles thus formed are equal to one another.*

THEOREM 15.

(*Eucl. Ax.* 11.)

right angles are equal to one another.

A B C and D E F be two right angles.



the sides B C and E F be produced beyond the vertex in I ; then the lines A B and D E will be respectively perpen- on the lines I C and H F [*Def.* 11]. Now, let the two figures lied to each other so that the lines I C and H F coincide with ther, and so that the point B coincide with the point E; then the diculars A B and *D E will* coincide also [I. 9]: hence the angles A B C and D E F coincide with each other; therefore they are equal to each other, W. W. T. B. D. Inversely

If one of two unequal angles is a right angle, the other is not a right angle.

Corollary. *The complements and supplements of equal angles are equal to one another,* and conversely.

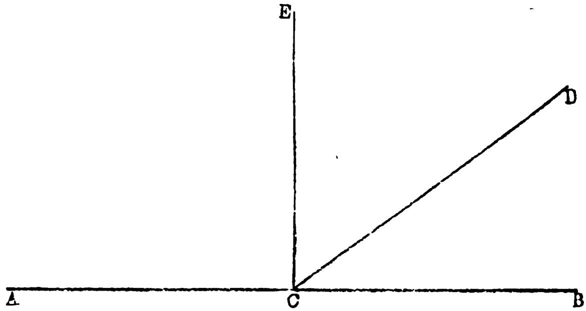
Scholium. The magnitude of the right angle being constant, it may be used advantageously as a unit for measuring all angles.

Theorem 16.

(*Eucl. I.* 13.)

Two adjacent angles formed by a straight line meeting another are together equal to two right angles.

Let the straight line C D meet another A B in the point C.



Let C E be a perpendicular on A B in the same point C. The angle E C B is composed of the angles E C D and D C B; hence the angles A C E, E C D and D C B are together equal to the angles A C E and E C B: but the angles A C E and E C D form together the angle A C D, and the angles A C E and E C B are two right angles [*Def.* 11]; therefore the angles A C D and D C B are together equal to two right angles, W. W. T. B. D.

Corollary I. *A straight angle is equal to two right angles.*

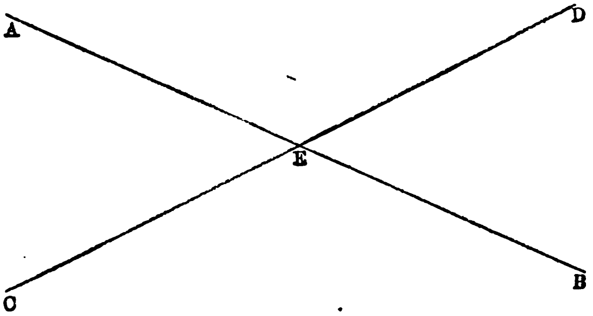
Corollary II. *If one of two adjacent angles formed by a straight meeting another is a right angle, the other is a right angle also.*

Corollary III. *All the angles made by any number of straight s meeting in one point on one side of a straight line, are together al to two right angles.*

Theorem 17.

If two straight lines cut each other, the four angles thus med are together equal to four right angles.

Let A B and C D be two straight lines cutting each other at E.



The angles A E D and A E C are together equal to two right angles [ ]; also the angles C E B and B E D are together equal to two ht angles: therefore the four angles A E D, A E C, C E B, and E D are together equal to four right angles [iv], W. W. T. B. D.

Corollary I. *An angle and its explement are together equal to r right angles.*

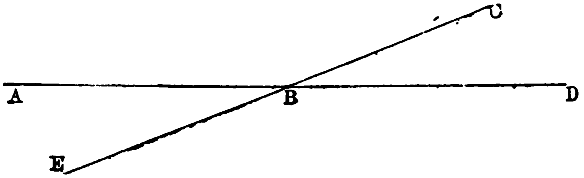
Corollary II. *All the angles formed by any number of straight es meeting in one point, are together equal to four right angles.*

THEOREM 18.

(*Eucl. I. 14.*)

If two adjacent angles, having the same vertex, be together equal to two right angles, the sides which are not common are the continuation of each other. (Conv. of 16.)

Let A B C and C B D be two adjacent angles equal together to two right angles and having the same vertex B.



Let B C be produced in E beyond the vertex B. Because the angles A B C and A B E are together equal to two right angles [16], and the angles A B C and C B D are also together equal to two right angles [*Hyp.*], the angles A B C and A B E are together equal to the angles A B C and C B D [i]; hence, the angle C B D is equal to the angle A B E [viii]. Likewise the angle A B C is equal to the angle E B D; hence the angles A B C and C B D are together equal to the angles A B E and E B D, or the angle A B D is equal to its explement: therefore the angle A B D is straight [5]; that is, A D is a straight line, W. W. T. B. D.

COROLLARY. *The angle equal to two right angles is straight.*

SECTION III.

ON THE COMPARISON OF ANGLES.

Each angle of the two groups of angles formed by two straight lines and a transversal, has received a different name when taken with one of the other group.

13. Two angles, that is, one of each group of angles

formed by two straight lines and a transversal, taken on the same side of the transversal, are *interior* angles if they are both inside the two straight lines; *exterior* angles if they are both outside the two straight lines; and *corresponding* angles if they are one inside and the other outside the two straight lines.

14. Two angles, that is, one of each group of angles formed by two straight lines and a transversal, taken one on each side of the transversal, are *internally alternate,* or simply *alternate*, when they are both inside the two straight lines; and *externally alternate* when they are both outside the two straight lines.

THEOREM 19.

Two angles the sides of which are the symmetrics of each other, each of each, are equal to one another.

For, if the plane containing them be folded upon the line of symmetry, their sides will coincide and their vertices also; they will then form one and the same angle: therefore they are equal to each other.

COROLLARY. *The angles made by symmetric lines with their line of symmetry, are equal to each other.*

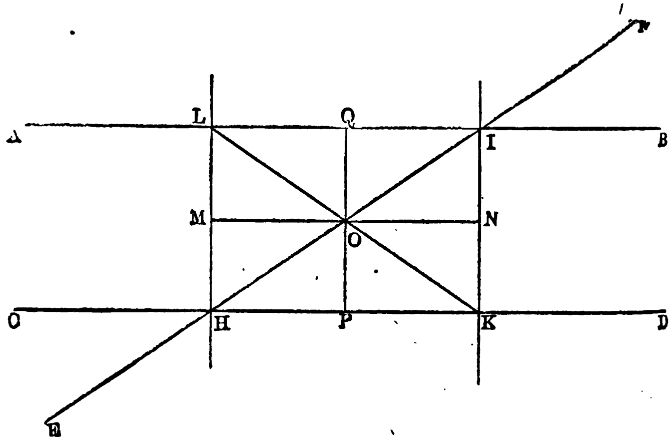
THEOREM 20.

(*Eucl. I.* 29.)

Alternate angles formed by two parallels and a transversal, are equal to each other.

Let A B and C D be two parallels cut by the transversal E F at H and I.

Let K I and H L be two perpendiculars to A B and C D [I. 17], through the points H and I, and let K L be joined, cutting H I in



O; let M N be the middle line of the intercepts H L and K I, and let P Q be the middle line of the intercepts H K and L I: then M N and P Q cut each other in O [I. 25 iii]. Because H K and L I are symmetric to M N [I. 33 ii], the lines I H and L K are symmetric to M N [I. 32 i]; hence the angles H I L and L K H are equal to each other [19]. Because H L and K I are symmetric to P Q [I. 33 ii], the lines H I and K L are symmetric to P Q [I. 32 i]; also H P and K P are symmetric to the same line [I. 27 ii]: therefore the angles I H P and L K P are equal to each other [19]. But the angles H I L and L K H are equal to each other [*Dem.*]; therefore the angles H I L and I H P, that is, the alternate angles E I A and F H D, are equal to each other [i]: now, their verticals and supplements, which are also alternate angles, are equal to each other; therefore the alternate angles formed by the two parallels A B and C D with the transversal E F, are equal to each other, W. W. T. B. D. Inversely *Two straight lines forming with a transversal unequal alternate angles, are not parallel to each other.*

ROLLARY I. *The corresponding angles formed by two parallels*
a transversal, are equal to each other.

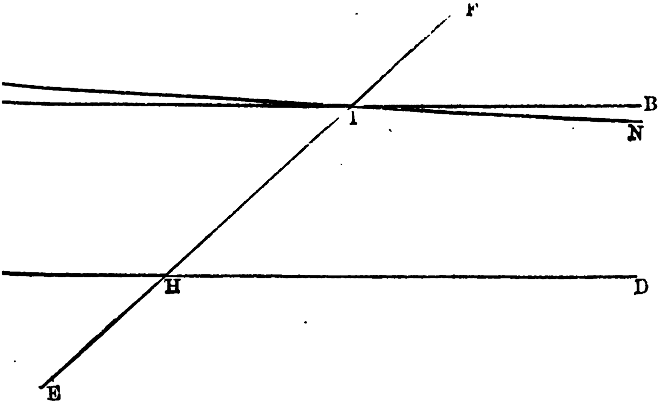
ROLLARY II. *The interior and exterior angles formed by two*
llels with a transversal, are supplementary.

THEOREM 21.

(*Eucl. I.* 27.)

onversely, *if two straight lines form with a transversal*
l alternate angles, they are parallel to each other.

et A B and C D be two straight lines cut by the transversal E F,
let the alternate angles H I B and I H C be equal to each other.



et M N be a parallel to C D, through the point I; then the alternate
es H I N and I H C are equal to each other [20]. Because the angle
B is equal to the angle I H C [*Hyp.*], the angles H I B and
N are equal to each other [i]; therefore the straight lines A B
M N coincide with each other: but M N is parallel to C D;
efore the line A B, which coincides with it, is also parallel to C D,
h W. T. B. D. Inversely
wo straight lines, not parallel to each other, form with a trans-
l, unequal alternate angles.

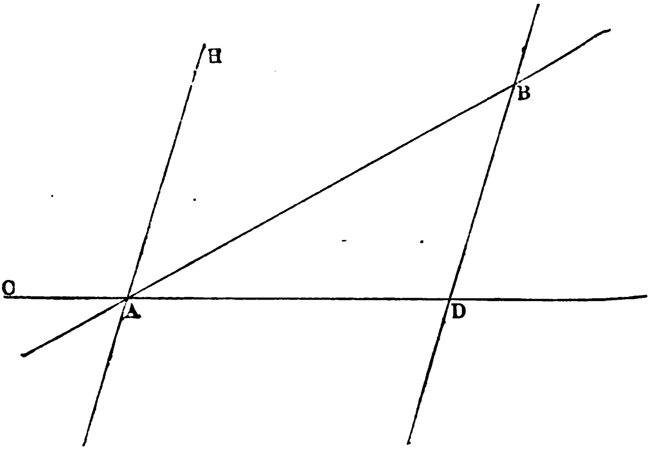
COROLLARY I. *If two straight lines form with a transversal equal corresponding angles, they are parallel to each other.*

COROLLARY II. *If two straight lines form with a transversal supplementary interior or exterior angles, they are parallel to each other.*

THEOREM 22.

If two straight lines cut each other, the interior angles formed by a transversal towards their intersection, are together supplementary to the angle formed by the straight lines towards the transversal.

Let A B and C D be two straight lines cutting each other in A and forming with the transversal B D the interior angles A B D and A D B.



Let A H be a parallel to D B through the point A. Then the alternate angles A B D and B A H, formed by the parallels A H and D B with the transversal A B, are equal to each other [20]; also corresponding angles A D B and C A H, formed by the same allels with the transversal C D, are equal to each other [20 i.]: refore the angles A B D and A D B are together equal to the les B A H and C A H, that is, to the angle B A C. But the le B A C is the supplement of the angle D A B; therefore the les A B D and A D B are together supplementary to the angle D, which W. T. B. D. Inversely

f two straight lines form on one side of a transversal, interior les which are together equal to, or greater than, two right angles, they not, however far produced, meet on the same side of the transversal.

OROLLARY I. *If two straight lines form on one side of a trans- sal interior angles which are together less than two right angles, the straight lines, indefinitely produced, shall meet on the same side of transversal* (I. 18 i).

OROLLARY II. *The straight line subtending a right or obtuse le forms towards the vertex two acute angles.*

OROLLARY III. *The straight line subtending a right angle forms ards the vertex two complementary angles.*

OROLLARY IV. *The bisectrices of the interior angles formed by two rallels with a transversal are perpendicular to each other.*

OROLLARY V. *If one of two angles subtended on the same side by same straight line, has its vertex in the angular space of the other, first is greater than the second.*

THEOREM 23.

Two angles the sides of which are parallel each to each, d have the same direction, are equal to one another.

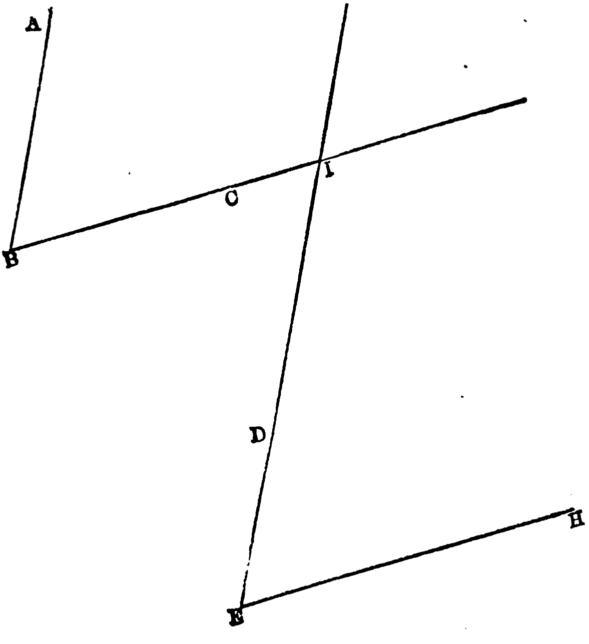
et A B C and D E H be two angles such that the sides B A and be parallel to one another, and also the sides B C and E H.

et the sides E D and B C be produced indefinitely until they each other in I [I. 18i]. Then the alternate angles A B C and E, formed by the two parallels A B and I E with the transversal

B I, are equal to each other [20]; also the alternate angles B I E and D E H, formed by the parallels B I and E H with the transversal I E, are equal to each other: therefore the angle A B C is equal to the angle D E H [i], which W. T. B. D.

Corollary I. *Two angles, the sides of which are parallel each to each and have opposite directions, are equal to each other.*

Corollary II. *If the sides of an angle be parallel each to each to those of another angle, one with the same direction and the other with the opposite direction, the two angles are supplementary to each other.*

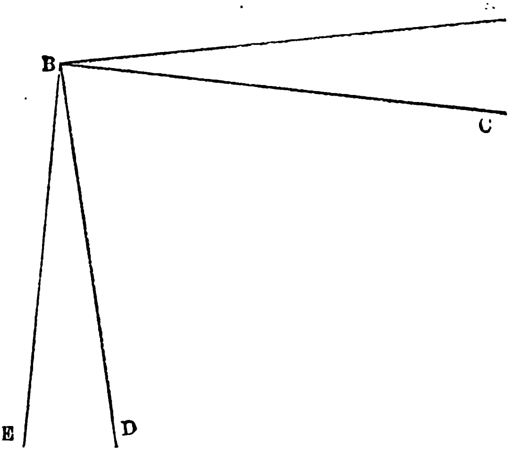


Scholium. Two angles the sides of one of which are parallel, each to each, to the sides of the other, are either equal or supplementary to each other.

THEOREM 24.

If two angles have the same vertex, and if the lines forming one of them be perpendicular on the same side of the lines forming the other, each to each, the two angles are equal to each other.

Let the angles A B C and D B E have the same vertex B; let also B D be perpendicular on B A, and B E perpendicular on B C.



The right angles A B D and C B E are equal to each other [15]: but they have in common the angle C B D; therefore the angles A B C and D B E, which are the parts not in common, are equal to each other [viii], W.W.T.B.D.

COROLLARY I. *If the lines forming two angles be perpendicular each to each, and if the parallels to the lines forming one of them, through the vertex of the other, be on the same side of the lines forming the latter, the two angles are equal to each other.*

COROLLARY II. *If the lines forming two angles be perpendicular, each to each, and if the parallels to the lines forming one of them, through the vertex of the other, be not on the same side of the lines forming the latter, the two angles are supplementary to each other.*

Scholium. Two angles the sides of one of which are perpendicular, each to each, to the sides of the other, are either equal or supplementary to each other.

SECTION IV.

ON CENTRE ANGLES.

DEFINITIONS.

15. A *centre* angle, or angle *at the centre*, is that which is formed by two radii of a circumference; the arc intercepted between the sides of the angle is its *arc;* the chord subtending the arc also *subtends* the angle; and the angle *stands* upon its arc. In the same circumference, or equal circumferences, *complementary* arcs are those of a centre angle and its complement; *supplementary* arcs are those of a centre angle and its supplement; *explementary* arcs are those of a centre angle and its explement.

16. An *inscribed* angle, or angle *at the circumference,* is that which is formed by two chords meeting at a point of the circumference; the arc intercepted between the sides of the angle is its *arc;* the chord subtending the arc *subtends* the angle; and the angle *stands* upon its arc.

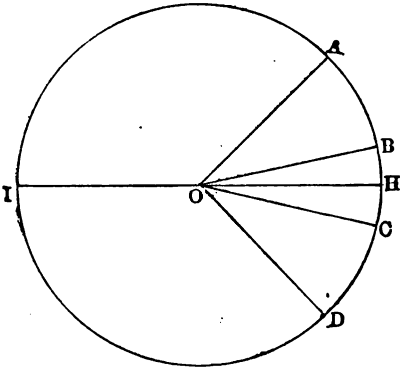
17. A *circumscribed* angle is that which is formed by two tangents to the same circumference; the arcs determined by the points of tangence, are its *arcs:* the arc convex to its vertex is its *convex* arc; and the arc concave to its vertex is its *concave* arc. A circumscribed angle is said to be circumscribed *at* the points of tangence of its *sides.*

THEOREM 25.

(*Eucl. III.* 26.)

In the same circumference, the arcs of equal centre angles are equal to one another.

Let A O B and C O D be two equal centre angles in the circumference O, and let the radius O H be the bisectrix of the angle B O C.



Let the bisectrix O H be produced beyond the vertex to a point I of the circumference: then I H is a diameter. Let the plane containing the figure be folded upon I H; then the two halves of the circumference O will coincide with each other [I. 39]. Because the two angles H O B and H O C are equal to each other [*Const.*], O B and O C will coincide with each other; also the point B will coincide with the point C. Because the angles A O B and C O D are equal to each other [*Hyp.*], the point A will coincide with the point D: then the arcs A B and C D will have the same extremities; therefore they are equal to each other, W. W. T. B. D. Inversely

In the same circumference, centre angles standing upon unequal arcs are unequal.

COROLLARY I. *In equal circumferences, the arcs of equal centre angles are equal to one another.*

COROLLARY II. *The alternate arcs determined by two diameters, are equal to each other.*

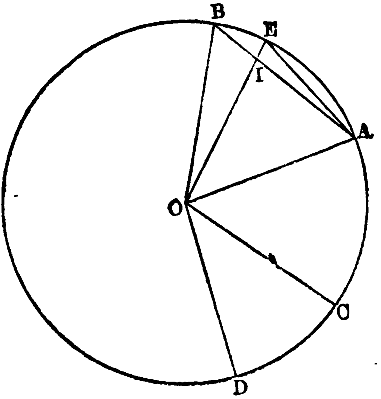
COROLLARY III. *Two diameters perpendicular to each other divide the circumference into four equal parts.*

THEOREM 26.

(*Eucl. III.* 27.)

In the same circumference, the arcs of unequal centre angles are unequal.

Let A O B be a centre angle greater than another C O D in the same circumference O.



Let the centre angle A O E be equal to the angle C O D. Because the angle A O B is greater than the angle C O D, it will be greater than the angle A O E, and the difference in direction between O A and O B will be greater than the difference in direction between O A and O E; therefore the radius O E will have a position between O A and O B. Let A B and A E be joined, and let I be the point where A B cuts the radius O E. The straight line A E is less than A I and I E together [I. 6 i]; also O B is less than O I and I B together: therefore A E and O B are together less than A I, I E, O I, and I B [v], or less than A B and O E. Because O B and O E are two radii, they are equal to each other; thence A E is less than A B [v]: therefore the arc A E is less than the arc A B [I. 43 ii]. But the arcs C D and A E are equal to each other [25]; therefore the arc C D is less than the arc A B, which W. T. B. D. Inversely

In the same circumference, centre angles standing upon equal arcs are equal to each other. (Conv. of 25.)

COROLLARY I. *In equal circumferences, centre angles standing upon equal arcs are equal to one another.*

COROLLARY II. *In the same circumference, or in equal circumferences, a greater angle stands upon a greater arc,* and conversely.

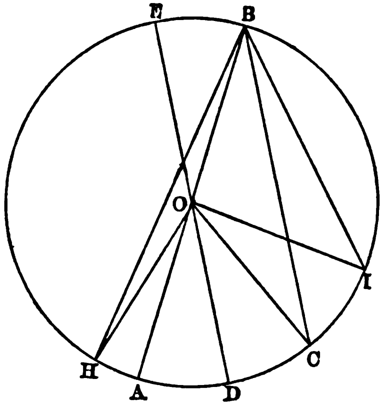
COROLLARY III. *The radius perpendicular to a chord, is the bisectrix of the centre angle subtended by the chord.*

THEOREM 27.

(*Eucl. III.* 20.)

An inscribed angle is equal to half the centre angle standing upon the same arc.

One of the chords forming the inscribed angle may pass through the centre of the circumference, or there may be one chord on each side of the centre, or both chords may be on the same side of the centre.



Let A B C be an angle inscribed in the circumference O, and let one of its sides A B pass through the centre.

Let O C be joined, and let the diameter E D be the bisectrix of the angle A O C. Because the vertical centre angles E O B and A O D are equal to each other [3], the arcs E B and A D are equal to each other [25]. Because also the centre angles A O D and D O C are equal to each other [*Const.*], the arcs A D and D C are equal to each other; therefore the arc E B is equal to the arc D C [i], and the chords E D and B C are parallel to one another [I. 53 i]. Because the corresponding angles A O D and A B C are formed by the parallels E D and B C, they are equal to each other [20 i]; but the angle A O D is half the angle A O C [*Const.*]: therefore the angle A B C is equal to half the angle A O C.

Let H B C be an angle inscribed in the circumference O, and let the centre O be between its sides.

Because the angle A B C is equal to half the angle A O C and, the angle H B A is equal to half the angle H O A [*Dem.*], the angles A B C and H B A are together equal to half the angles A O C and H O A together [ii]: that is, the angle H B C is equal to half the angle H O C.

Lastly, let C B I be an angle in the circumference O, and let the centre O be outside the angle.

Let O I be joined. Because the angle A B C is equal to half the angle A O C, and the angle A B I is equal to half the angle A O I [*Dem.*], the difference of the angles A B I and A B C is equal to half the difference of the angles A O I and A O C [ii]: that is, the angle C B I is equal to half the angle C O I.

Therefore, any inscribed angle is half the centre angle standing upon the same arc, W. W. T. B. D.

COROLLARY I. *An inscribed angle is equal to half the centre angle standing upon an equal arc,* and conversely.

COROLLARY II. *In the same circumference or in equal circumferences, all inscribed angles standing upon the same arc or equal arcs, are equal to one another,* and conversely.

COROLLARY III. *In the same circumference or in equal circumferences, a greater inscribed angle stands upon a greater arc,* and conversely.

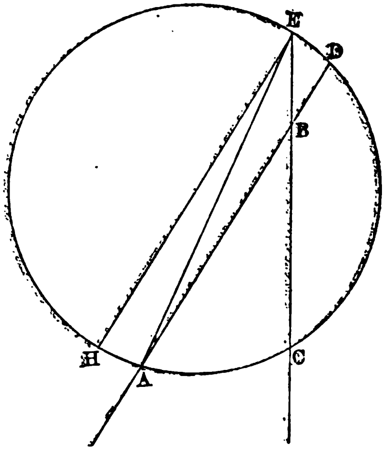
THEOREM 28.

An angle the vertex of which is within the circumference, is equal to the sum of the inscribed angles standing upon its arc and that of its vertical.

Let A B C be an angle the vertex of which is within the circumference A D C.

Let its sides be produced beyond the vertex to the points E and D of the circumference; let E H be a chord parallel to A D through the point E, and let A E be joined. The corresponding angles A B C and H E C formed by the parallels H E and A D, are equal to each other [20 i]; that is, the angle A B C is equal to the angles A E C and

A E H together. Because the arcs A H and D E, intercepted between the parallels H E and A D, are equal to each other [I. 53], the inscribed angles A E H and E A D are equal to each other [27 ii]; therefore the angle A B C is equal to the angles A E C and E A D which stand upon its arc and that of its vertical, W. W. T. B. D.



Corollary. *If an angle be formed by two secants, and if its vertex be without the circumference, it is equal to the difference of the inscribed angles standing upon the two arcs intercepted by its sides.*

Scholium. A centre angle is equal to twice the inscribed angle standing upon its arcs, or that of its vertical (27).

THEOREM 29.

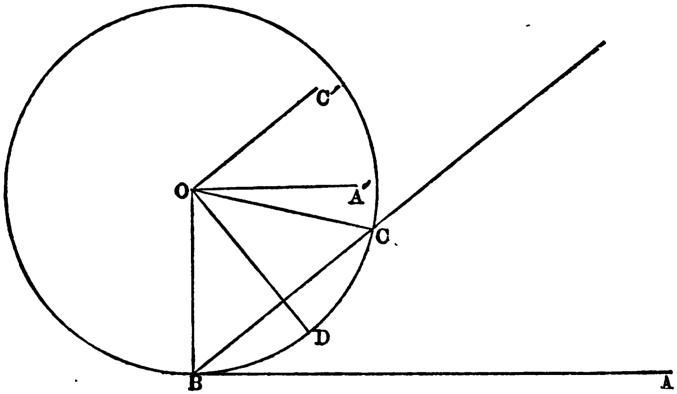
(*Eucl. III.* 32.)

The angle formed by a tangent and a chord terminating at the point of tangence, is equal to half the centre angle subtended by the chord.

Let A B C be an angle formed by a tangent A B to the circumference O, and the chord B C terminating at the point of tangence.

Let B O C be a centre angle standing upon the arc B C and let O D be a radius perpendicular on B C: then O D is the bisectrix of the angle B O C [26 iii]. Because the lines forming the angles A B C and B O D are perpendicular each to each, and the parallels O A′ and O C′ to B A and B C through the vertex O, are on the same side of

O B and O D, the angles A B C and B O D are equal to each other [24 i]; but the angle B O D is half the angle B O C; therefore



the angle A B C is equal to half the centre angle B O C, which W. T. B. D.

Corollary I. Conversely, *the straight line which forms at the extremity of a chord subtending a centre angle, on the side opposite to the vertex, an angle equal to half the centre angle, is tangent to the circumference.*

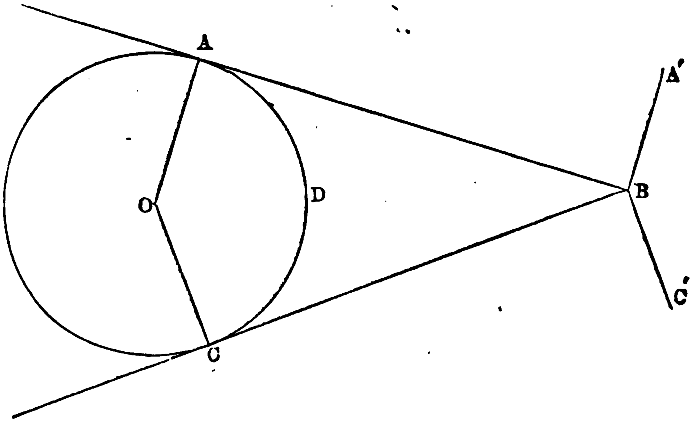
Corollary II. *The angle formed by a tangent and a chord terminating at the point of tangence, is equal to the inscribed angle subtended by the chord,* and conversely.

Theorem 30.

A circumscribed angle is supplementary to the centre angle standing upon its convex arc.

Let A B C be a circumscribed angle, and let A and C be the points of tangence of its sides; let A O C be a centre angle standing upon the convex arc A D C.

Because the radii forming the centre angle A O C are perpendicular to the tangents forming the angle A B C, each to each [I. 55], and because the parallels B A′ and B C′ to O A and O C through the vertex B are not on the same side of B A and B C, the angles A B C and



A O C are supplementary to each other [24 ii]; therefore the circumscribed angle A B C is supplementary to the centre angle standing upon its convex arc, W. W. T. B. D.

COROLLARY I. *The angles formed towards the vertex by a straight line subtending a circumscribed angle, are together equal to the centre angle standing upon the convex arc* (22).

COROLLARY II. *Each of the angles formed towards the vertex of a circumscribed angle by the chord of the convex arc, is equal to half the centre angle standing upon the same arc.*

COROLLARY III. *Two angles circumscribed to the same circumference and having equal convex or concave arcs, are equal to each other.*

COROLLARY IV. *A greater circumscribed angle has a greater concave arc, or a smaller convex arc,* and conversely.

Scholium. The angles formed towards the vertex by a tangent subtending a circumscribed angle are together supplementary to that angle.

SECTION V.

ON THE MEASUREMENT OF ANGLES.

DEFINITIONS.

If it be supposed that the vertices of all angles, are the centres of the same number of equal circumferences (that is, circumferences of equal radii), then the arcs of all equal angles will be equal to each other [25], and conversely all angles standing upon equal arcs will be equal to each other [26]. Thus all angles standing upon an arc of one degree will be equal to one another: each of them is called an angle *of one degree.* As the degree is generally used as a unit to measure all arcs, so the angle of one degree is most generally used as a unit to measure all angular magnitudes.

An angle and an arc which bear the same relation to their respective units, are evidently of the same value [*Gen. Def.* 15]; each is called the *measure* of the other. Hence,

18. An angle and an arc are said to be the *measure* of each other, when they have the same value.

Thus the measure of an angle of one degree is the arc of one degree.

It is obvious that angles which have the same measure are equal to one another [26], and conversely, that arcs which are the measure of equal angles are equal to each other [25].

THEOREM 31.

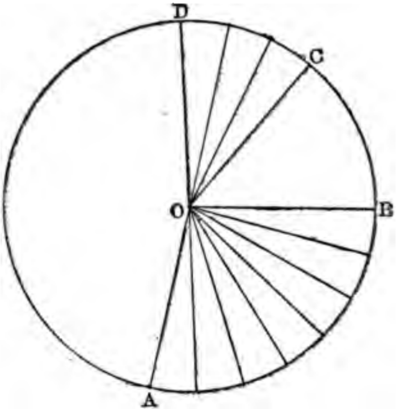
(*Eucl. VI.* 33.)

In the same circumference, two centre angles are to each other as their arcs.

Let A O B and C O D be any two centre angles in the circumference O.

Let the arcs A B and C D be divided into equal parts, small enough to be contained an exact number of times in each; let, for instance, A B contain *seven*, and C D, *three* of these parts: the arc A B is then to the arc C D as *seven* is to *three* [x].

Let there be a radius to each point of division of both arcs. Because all the centre angles thus formed stand upon equal arcs [*Const.*], they are equal to one another [25]; therefore the angle A O B is to the angle C O D as *seven* is to *three* [x].



Therefore, the angle A O B is to the angle C O D as the arc A B is to the arc C D [i].

Should the arcs AB and C D be incommensurable, let them be imagined to be divided into their infinitely small elements; and through each point of division, let a radius be imagined. The two angles A O B and C O D will be divided into a number, finite or infinite, of centre angles standing upon equal arcs [*Post.* IX]; therefore these partial angles will be all equal to one another [25]. However small these partial angles may be, it is evident that the number of the angles, will be equal to that of the arcs in each of the angles A O B and C O D; therefore they are still as their arcs, W. W. T. B. D.

COROLLARY I. *In equal circumferences, two centre angles are to each other as their arcs.*

COROLLARY II. *In the same circumference or in equal circumferences two inscribed or circumscribed angles are to each other as their arcs.*

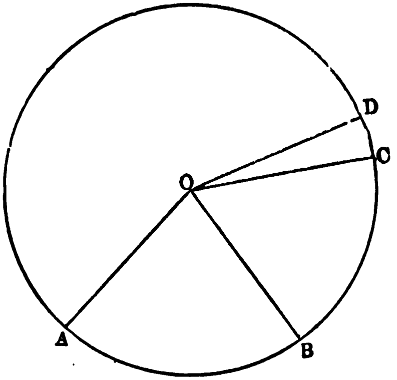
Scholium. Two angles are to each other as the arcs described with the same radius between their sides, and with their vortices as centres.

THEOREM 32.

The measure of a centre angle is its arc.

Let A O B be any centre angle and let C O D be the angular unit.

Let the arc C D be the unit of arc. Because the angle A O B is to the angle C O D as the arc A B is to the arc C D [31], the ratio of the angle A O B to the angular unit is the same as that of the arc A B to the unit of arc: that is, their value is the same [*Gen. Def.* 15]; therefore the arc A B is the measure of the angle A O B [*Def.* 18], which W. T. B. D.



COROLLARY I. *The measure of the right angle is the fourth part of the circumference*, and conversely.

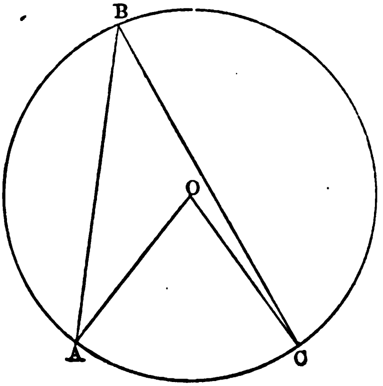
COROLLARY II. *The measure of a straight angle is half the circumference*, and conversely.

Scholium. The right angle is an angle of 90 degrees.

THEOREM 33.

The measure of an inscribed angle is half of its arc.

Let A C be the arc of an angle A B C inscribed in the circumference O.



Let the centre O be joined to the points A and C. Because the arc A C is the measure of the angle A O C [32], half the arc A C is the measure of half the angle A O C [vii]; but the angle A B C is equal to half the angle A O C [27]: therefore half the arc A C is also the measure of the angle A B C [i], which W. T. B. D.

COROLLARY I. *An angle standing upon a semicircumference is a right angle.*

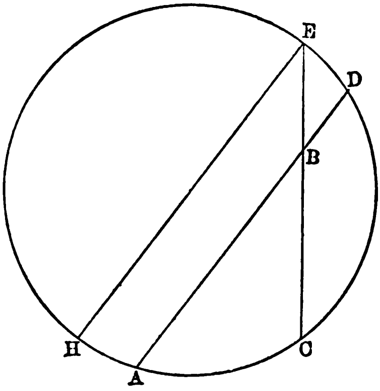
COROLLARY II. *An angle standing upon a circular arc is an acute angle, and an angle standing upon a semicircular arc is an obtuse angle.*

COROLLARY III. *Angles standing upon opposite arcs are supplementary to each other.*

COROLLARY IV. *The circumference described on a straight line which subtends a right angle, passes through the vertex.*

THEOREM 34.

The measure of an angle the vertex of which is within the circumference, is half its arc together with half that of its vertical.



Let A B C be an angle the vertex of which is within the circumference.

Let its sides be produced beyond the vertex to D and E, and let E H be a parallel to A D. Because the corresponding angles A B C and H E C are equal to each other [20 i], and half the arc H C is the measure of the angle H E C [34], half the arc H C is also the measure of the angle A B C [i]. But the arc HC is composed of the arcs A C and H A, and the arc H A is equal to the arc E D [I. 53]; therefore half the arc A C together with half the arc E D form the measure of the angle A B C, which W. T. B. D.

COROLLARY I. *If an angle be formed by two secants, and its vertex be* *without the circumference, its measure is half the difference of the arcs intercepted between its sides.*

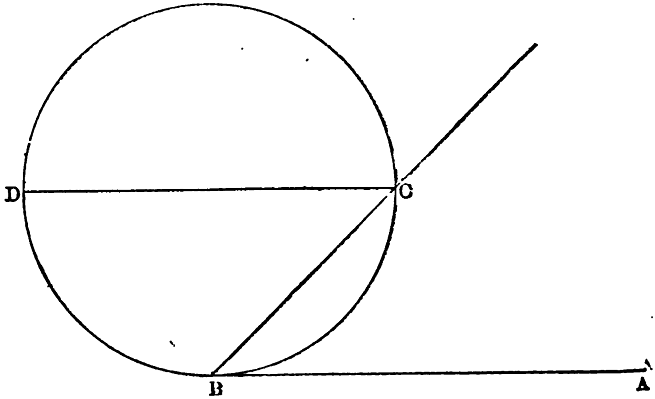
Corollary II. *If the vertices of two supplementary angles subtended by the same straight line, be on opposite sides of the latter, the vertices and the extremities of the subtending line are points of the same circumference.*

Corollary III. *If two chords of a circumference cut each other perpendicularly, the arcs of two vertical angles thus formed, are together equal to a semicircumference.*

Theorem 35.

The measure of an angle formed by a tangent and a chord terminating at the point of tangence, is half the arc subtended by the chord.

Let A B C be an angle formed by a tangent A B to the circumference B C D, and the chord B C terminating at the point of tangence B.



Let C D be a chord parallel to A B. Because the alternate angles A B C and B C D are equal to each other [20], and half the arc B D is the measure of the angle B C D [33], half the arc B D is also the measure of the angle A B C. But the arcs B C and B D are equal to each other [I. 55 iii]; therefore, half the arc B C is the measure of the angle A B C, which W. T. B. D.

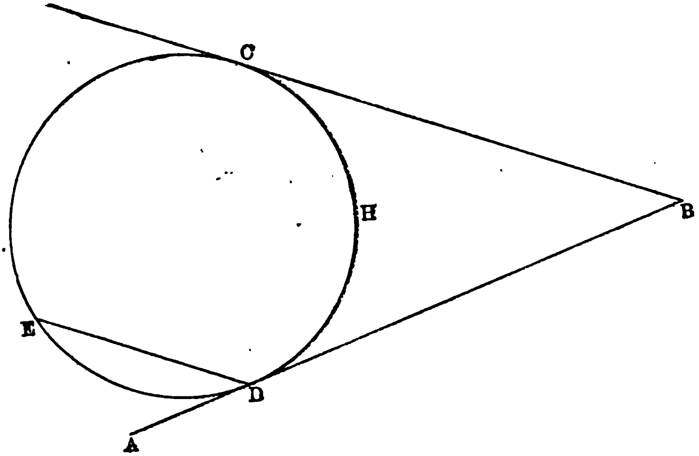
COROLLARY. Conversely, *if the measure of the angle formed by a chord and a straight line passing through its extremity be equal to half the arc subtended by the chord, the straight line is tangent to the circumference.*

Scholium. The measure of the angle formed by a tangent and the diameter terminating at the point of tangence, is the fourth part of the circumference (I. 55).

THEOREM 36.

The measure of a circumscribed angle is half the difference of its arcs.

Let A B C be an angle formed by the two tangents B A and B C, and let C and D be the points of tangence.



Let D E be a chord parallel to B C through the point D. Because the measure of the angle A D E is half the arc D E [35], and the corresponding angles A D E and A B C are equal to each other [20], the measure of the angle A B C is also half the arc D E [i]. But the arcs E C and D C are equal to each other [I. 53]; the arc D E is then equal to the difference of the arcs C E D and C H D; therefore the measure of the angle A B C is half the difference of the arcs C E D and C H D, which W. T. B. D.

COROLLARY. *The measure of an angle formed by a tangent and a secant, the vertex of which is without the circumference, is half the difference of the arcs intercepted by its sides.*

Scholium. The angle formed by two straight lines tangent at the extremities of a diameter is nought (I. 55 ii).

CHAPTER II.

ON PROPORTIONAL LINES.

SECTION I.

ON STRAIGHT LINES CUT BY PARALLELS.

DEFINITIONS.

19. Straight lines are *proportionally*, or *similarly*, divided hen any two adjoining parts of one of them are proportonal to two adjoining parts of each of the others; the oints of division separating the homologous parts are called *mologous* points.

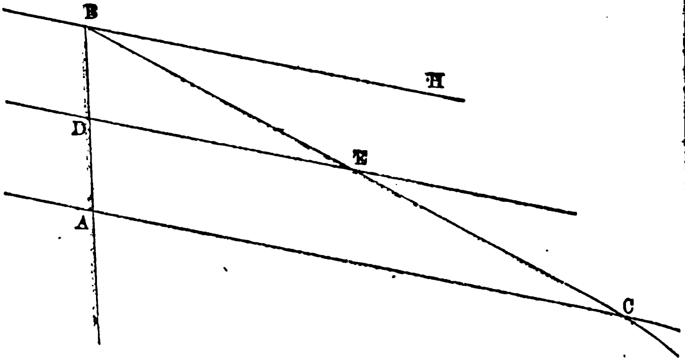
20. When the sides of an angle are cut proportionally by raight lines, the portions of the latter intercepted between he sides, subtend the angle *proportionally*.

THEOREM 37.

If two parallels which cut both sides of an angle, determine n one side two equal portions thereof, the portions of the other de are equal to each other.

Let A B C be an angle the sides of which are cut by the parallels E and A C, and let the portions B D and D A determined by the arallels on the side B A, be equal to each other.
Let B H be a parallel to D E and A C [I. 17 iii], through the point

E. Because the intercepts of B A between the three parallels B H, D E and A C are equal to each other [*Hyp.*], the parallels B H and



A C are equally distant from the line D E [I. 24]: therefore the intercepts of B C between the same parallels, are equal to each other [I. 23], W. W. T. B. D. Inversely

If two parallels which cut both sides of an angle determine on one side two unequal portions thereof, the portions determined on the other side are unequal.

Corollary. *If the sides of an angle be cut by any number of parallels, and the portions determined on one side by the parallels be equal to one another, the portions determined on the other side are also equal to one another.*

Theorem 38.

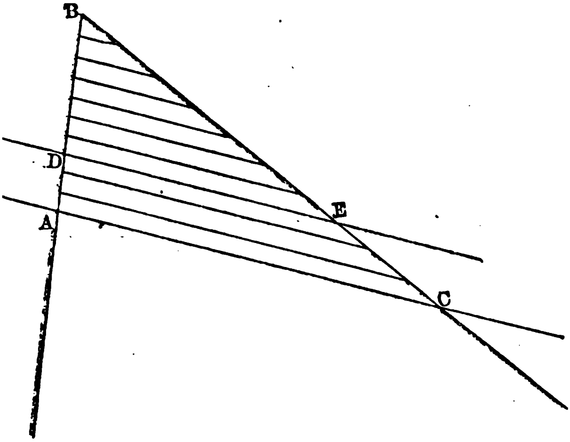
(*Eucl. VI.* 2.)

The sides of an angle are cut proportionally by two parallels.

Let there be an angle A B C, the sides of which are cut by the parallels D E and A C.

Let the portions B D and D A of the line A B be divided into equal parts, small enough to be contained an exact number of times in

t, for instance, B D contain *seven* and D A *three* of these
hen B D will be to D A as *seven* is to *three* [x].



let there be a parallel to D E through each point of division
then B C will be divided into equal parts by these parallels
B E will contain *seven* of these parts and E C *three* of them:
e B E will be to E C as *seven* is to *three* [x].
fore, B D is to D A as B E is to E C [i].
d the lines B D and D A be incommensurable, let them be
d to be divided into their infinitely small elements, which are
e for both lines [*Post.* IV], and through each point of division,
allel to D E be imagined; then B C will also be divided into a
, finite or infinite, of parts equal to one another [37 i]. How-
all these parts may be, it is evident that there will be the
mber of them in B E as in B D, and the same number in E C
A; therefore B D will be still to D A, as B E is to E C: that
portions determined by the parallels on the sides of the
re proportional, W. W. T. B. D.
LARY I. *The sides of vertical angles are cut proportionally by*
llels.

COROLLARY II. *Any number of parallels cut proportionally the sides of an angle, or vertical angles.*

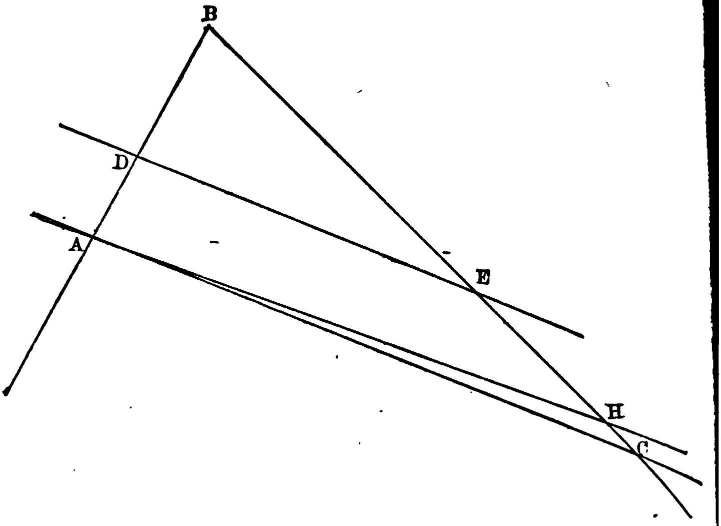
COROLLARY III. *The intercepts of two transversals between three parallels, are proportional.*

THEOREM 39.

(*Eucl. VI.* 2.)

Conversely, *two straight lines which cut proportionally the sides of an angle, are parallel to each other.*

Let the angle A B C be cut proportionally by the straight lines A C and D E, so that E C be to B E as D A is to B D.



Let A H be a parallel to D E, through the point A. Then E H is to B E as D A is to B D [38]. Because E C is to B E as D A is to B D [*Hyp.*], the line E H is equal to E C [xvii]: therefore, the points H and C are one and the same point. Because the straight lines A C and A H have two points in common, they coincide throughout with each other

[I. 4]: now, A H is parallel to D E [*Const.*]; therefore A C is also parallel to D E, which W. T. B. D.

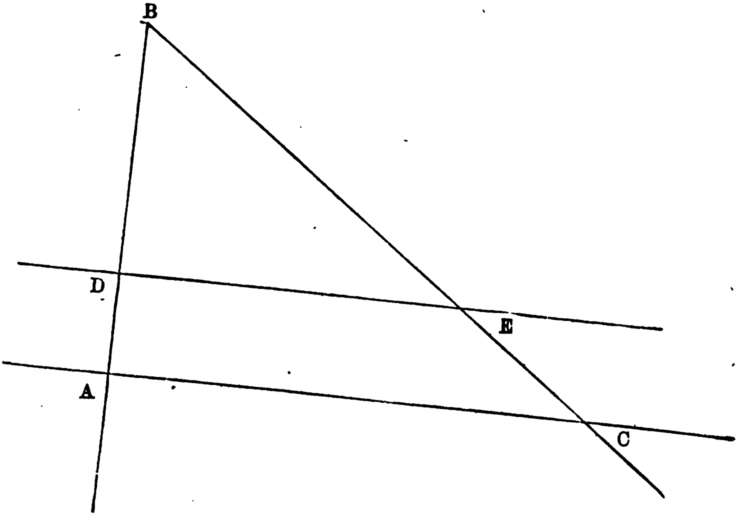
COROLLARY. *Two straight lines which cut proportionally the sides of vertical angles, are parallel to one another.*

Scholium. Two straight lines which determine equal parts of each of the sides of an angle, or vertical angles, are parallel to each other (I. 26 v).

THEOREM 40.

If the sides of an angle be cut by parallels, the distances from the vertex to the points of section are proportional.

Let the sides of an angle A B C be cut by the parallels D E and A C.



Because B D is to D A as B E is to E C [38], the lines B D and D A together are to B E and E C as B D is to B E [xv]; that is, B A is to B C as B D is to B E: therefore B A is to B D as B C is to B E [xiii]; which W. T. B. D.

COROLLARY I. Conversely, *if two straight lines cut the side of an angle, and if the distances from the vertex to the points of section be proportional, the straight lines are parallel to each other.*

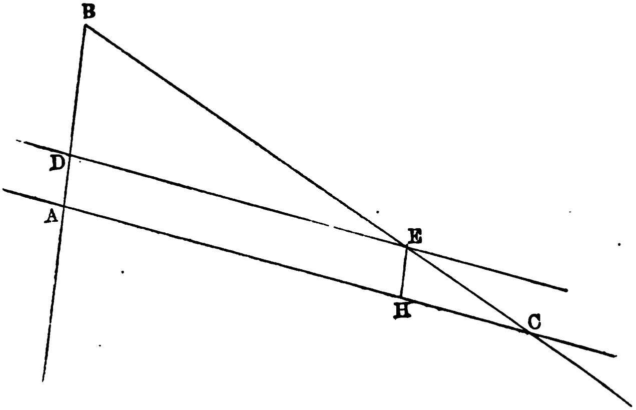
COROLLARY II. *If the sides of vertical angles be cut by parallels, the distances from the vertex to the points of section are proportional,* and conversely.

COROLLARY III. *If the sides of an angle, or vertical angles, be cut by parallels, the distances from the vertex to the points of section are proportional to the distances from the vertex to the intersecting parallels,* and conversely.

COROLLARY IV. *If the sides of an angle, or vertical angles, be cut by parallels, the distances from the vertex, of the points of section of each parallel, are proportional to those of any of the others,* and conversely.

THEOREM 41.

The intercepts of two parallels between the sides of an angle are proportional to the distances from the vertex to their points of intersection with one side.



Let A B C be an angle the sides of which are cut by the parallels D E and A C.

Let E H be parallel to B A. Then A H is to A C as B E is to B C [40]; but the intercepts D E and A H between the parallels E H and B A are equal to each other [I. 22]: therefore D E is to A C as B E is to B C, which W.T.B.D.

Corollary I. *If two parallels subtend an angle, and if the portions which they determine on one side be equal, the subtending line nearer to the vertex is equal to half the other.*

Corollary II. *The intercepts of two parallels between the sides of vertical angles, are proportional to the distances from the vertex to their points of intersection with one side and its prolongation.*

Corollary III. *If two parallels subtend an angle and its vertical, and if the portion which they determine on one side and its prolongation, be equal, these lines are equal to each other.*

Corollary IV. *The intercepts of any number of parallels between he sides of an angle, or vertical angles, are proportional to the distances from the vertex to their points of intersection with one side, r its prolongation.*

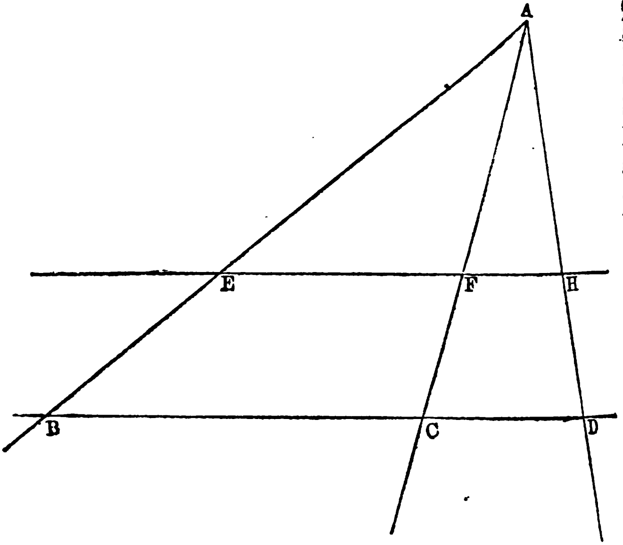
Corollary V. *The intercepts of parallels between the sides of an ngle, or vertical angles, are proportional to their distances from the ertex.*

Theorem 42.

Two parallels are cut proportionally by three transversals rawn from the same point.

Let the transversals A B, A C, and A D, which cut the straight ie B D in B, C and D, also cut its parallel E H in E, F d H.

Because the lines E F and B C are portions of two parallels interpted between the sides of the angle B A C, the line E F is to B C A F is to A C [41]; also, because F H and C D are portions of the same parallels intercepted between the sides of the angle C A D,



the line F H is to C D as A F is to A C: therefore, E F is to B C as F H is to C D [i], and E F is to F H as B C is to C D [xiii], which W. T. B. D.

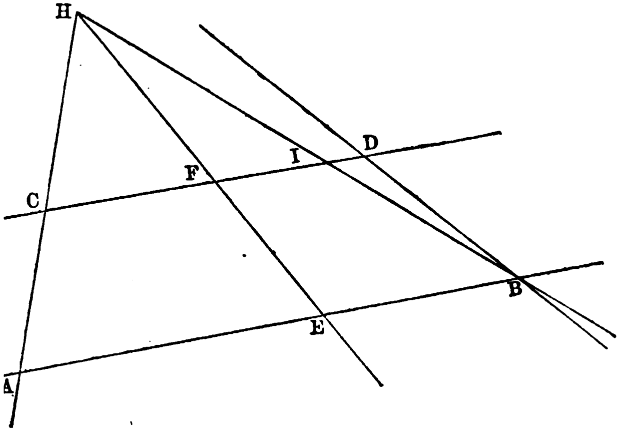
Corollary I. *If two parallels be cut by transversals drawn from one point, and if the intercepts of one of them between the transversals, be equal to one another, the intercepts of the other are equal to one another.*

Corollary II. *Parallels are cut proportionally by any number o transversals drawn from the same point.*

Theorem 43.

Conversely, *if the straight lines which pass through t homologous points of two parallels divided proportionally, me each other, they meet at the same point.*

A B and C D be two parallels divided by the transversals A C,
nd B D, so that F D be to C F as E B is to A E.



H be the meeting point of A C and E F and let H B be joined,
g C D in a point I. Because F I, C F, E B and A E are the
epts of two parallels between three transversals from the same
the line F I is to C F as E B is to A E [42]; but F D is to C F
is to A E [*Hyp.*]: therefore F I is equal to F D [xvii]; that is,
ints D and I are one and the same point. Because the straight
B D and B I have two points in common, they coincide through-
eir extension [I. 4]; therefore B D meets A C and E F in the
H, which W. T. B. D.

olium. The straight lines passing through the extremities of
nequal straight lines having a common middle perpendicular,
the latter in the same point (I. 30).

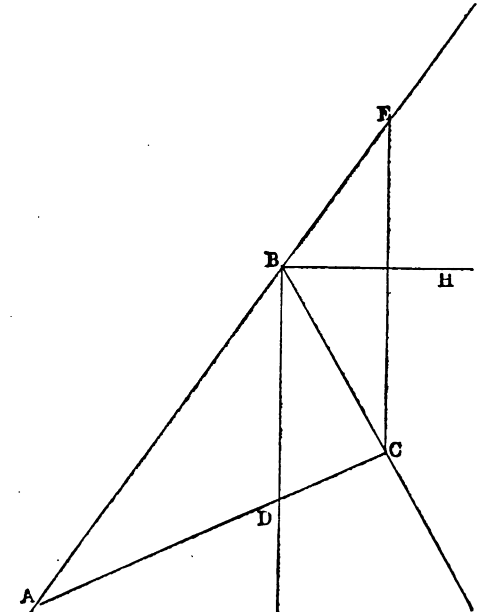
THEOREM 44.

(*Eucl. VI.* 3.)

e straight line subtending an angle is divided by the *bisectrix into portions, proportional to the distances from the vertex to their intersections with the sides.*

Let A C be a straight line subtending an angle A B C and let B D be the bisectrix of the angle, cutting A C in D.

Let the side B A of the angle be produced beyond the vertex, and let B H be the bisectrix of the adjacent angle C B E thus formed; let also C E be a parallel to D B through the point C: then the parallels D B and C E cut proportionally the sides of the angle C A E [38]; that is A D is to D C as A B is to B E. Because B H is perpendicular to B D [13 i], and E C is parallel to B D [*Const.*], the line B H is perpendicular to E C [I. 17], and E C subtends symmetrically the angle C B E [7 iii], that is, B E is equal to B C: but A D is to D C as A B is to B E [*Dem.*]; therefore A D is to D C as A B is to B C, which W. T. B. D.



Corollary I. Conversely, *if the straight line which subtends an angle, be divided into portions proportional to the distances from the vertex to their intersections with the sides, the straight line joining the point of section to the vertex is the bisectrix of the angle.*

COROLLARY II. *If a straight line meet another, the prolongation a straight line subtending one of the angle thus formed, to a oint of the bisectrix of the adjacent angle, and the subtending ne itself together with its prolongation, are proportional to the istances from the vertex to their intersections with the sides,* and onversely.

Scholium. The bisectrix of an angle bisects the straight line subending the angle symmetrically (7).

SECTION II.

ON STRAIGHT LINES IN THE CIRCUMFERENCE.

DEFINITIONS.

21. Two straight lines are in *medial* ratio when the greater is a mean proportional between their sums and the esser.

22. A straight line divided into two portions, is said to e *medially* divided, or divided *in extreme and mean ratio,* hen the two portions thereof are in medial ratio: the reater portion is called the *mean* of the straight line, and he lesser its *extreme.*

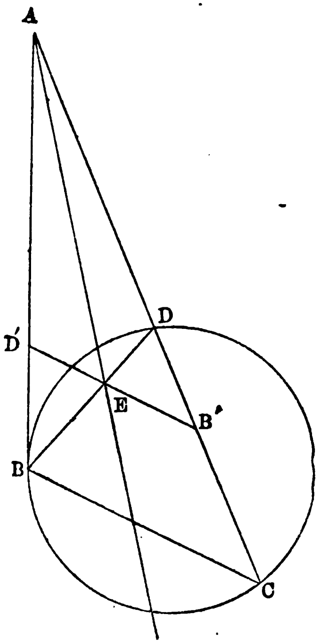
THEOREM 45.

If there be a tangent and a secant to a circumference from point without it, the tangent is a mean proportional to the cant and its outer part.

Let there be a tangent A B, and a secant A C of which the outer rt is A D.

Let A E be the bisectrix of the angle B A C; let A B′ be equal to

A B, and A D′, equal to A D; let also B C, B D and B′ D′ be joined.



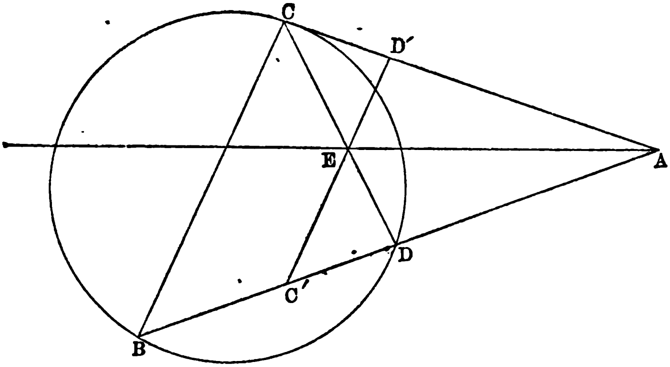
Because D and D′ are symmetric points of the angle B A C, they are symmetric to the bisectrix A E [6]; also B and B′ are symmetric to A E; thence B D and B′ D′ cut each other in a point E of A E [I. 32]. Because B E and B′ E are symmetric to A E, and because A B and A B′ are also symmetric to A E, the angle A B E is equal to the angle A B′ E [19]; because the measure of the inscribed angle B C D [33] is also the measure of the angle A B D formed by the tangent A B and the chord B D [35], the angle A B D or A B E is equal to the angle B C D: therefore the angle B C D or A C B is equal to the angle A B′ E [i]. Because the corresponding equal angles A C B and A B′ D′ are formed by the straight lines B C and D′ B′ with the transversal A C, the line D′ B′ is parallel to B C [21 i]; hence A C is to A B′ as A B is to A D′ [40]; but A B′ is equal to A B [*Const.*], and A D′, equal to A D; therefore A C is to A B as A B is to A D, which W. T. B. D.

COROLLARY. *If a point be joined to a circumference by a secant and a tangent equal to the chord of the latter, the secant is divided in extreme and mean ratio.*

THEOREM 46.

Conversely, *if there be a secant and another straight line drawn to a circumference from a point without it, and if* *the latter line be a mean proportional to the secant and its outer part, the straight line is tangent to the circumference.*

Let there be a secant, A B, of which the points of section are B and D, and the straight line A C drawn from the same point A to the same circumference; let also A B be to A C as A C is to A D.



Let A E be the bisectrix of the angle B A C; let A C′ be equal to A C, and A D′ equal to A D; let also B C, C D and C′ D′ be joined. Because D and D′ are symmetric points of the angle B A C, they are symmetric to the bisectrix A E [6]; also C and C′ are symmetric to A E: thence C D and C′ D′ cut each other in a point E of A E [I. 32]. Because C E and C′ E are symmetric to A E as also A C and A C′, the angle A C E is equal to the angle A C′ E [19]. Because A B is to A C as A C is to A D [*Hyp.*], and because A C′ and A D′ are equal to A C and A D respectively [*Const.*], A B is to A C′ as A C is to A D′; therefore D′ C′ is parallel to C B [40 i]. Because the corresponding angles A B C and A C′ D′ are equal to each other [20 i], and because the angle A C′ D′ is equal to the angle A C D [*Dem.*], the angles A C D and A B C are equal to each other [i]: but the chord C D, forming with

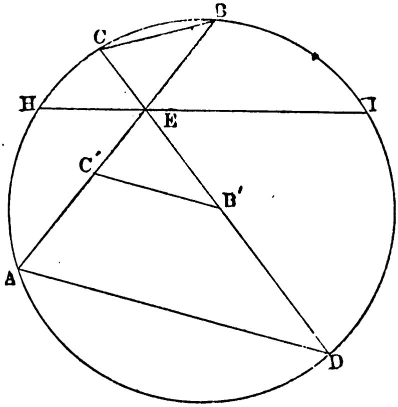
A C the angle A C D, subtends the inscribed angle A B C; therefore A C is tangent to the circumference B C D [29 ii], which W. T. B. D.

COROLLARY. *If there be a secant drawn to a circumference from a point without it, any straight line which is a mean proportional to the secant and its outer part, is equal to the tangent from the same point.*

Scholium. If there be a circumference through the extremities of the mean of a straight line, and if another straight line equal to the mean join the other extremity to the circumference, the latter line is a tangent to the circumference.

THEOREM 47.

The portions of two chords which cut each other, are reciprocally proportional.



Let there be two chords A B and C D, cutting each other in E.

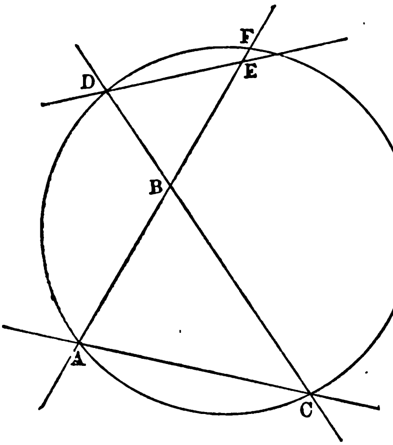
Let C B and A D be joined and let H I be the bisectrix of the angle A E C; let also E C′ be equal to E C, and E B′ equal to E B; and let C′ B′ be joined. Because B C is the symmetric of B′ C′ [6 iv], the line B E is the symmetric of B′E, and the angle E B′ C′ is equal to the angle E B C [19] or A B C: but the inscribed angles A D C and A B C are equal to each other [27 ii]; therefore the angle E B′ C′ is equal to the angle A D C [i]. Because the corresponding angles E B′ C′ and E D A are equal to each other, the lines B′ C′ and D A are parallel to each other [21 i]; then A E is to C′ E as D E is to B′ E [40]; but B′ E is equal to B E (*Const.*), and C′ E is equal to C E; therefore A E is to C E as D E is to B E, which W. T. B. D.

THEOREM 48.

ersely, *if the distances from the vertex of vertical angles*
tersections of the sides with two straight lines, be re-
ly proportional, the four intersections are points of the
cumference.

ere be two straight
C and D E cutting
of the vertical an-
C and D B E in the
, C, D and E; and
be to B C as B D
.

ere be a circumfer-
ugh the points A, C
utting the side B E
ecause A F and C D
hords cutting each
F is to B C as B D is
7]. But B E is to B C
to B A [*Hyp.*]; then
qual to B F [xvii]: that is, the points E and F are one and
point; therefore E is a point of the circumference A C D,
.T. B. D.

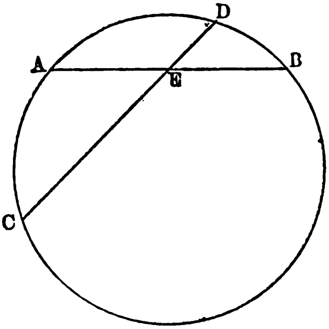


LARY. *If the distances from the vertex of vertical angles to the*
ons of the sides with two straight lines, be reciprocally pro-
, the angles formed by each line towards the vertex are equal,
ch, to those formed by the other.

THEOREM 49.

hord be bisected by another, each half of the first is
proportional to the two portions of the other.

Let a chord A B be bisected by a chord C D.



Because the chords A B and C D cut each other, C E is to A E as B E is to E D [47]; but A E is equal to E B [*Hyp.*]: therefore C E is to A E as A E is to E D, which W. T. B. D. Inversely

If one portion of a chord cutting another is not a mean proportional to the two proportions of the other, it is not equal to the other portion of the same chord.

COROLLARY I. Conversely, *if one portion of a chord cutting another, be a mean proportional to the two portions of the other, it is equal to the other portion of the same chord.*

COROLLARY II. *The perpendicular to a diameter, drawn from a point of the circumference, is a mean proportional to the two portions thereof.*

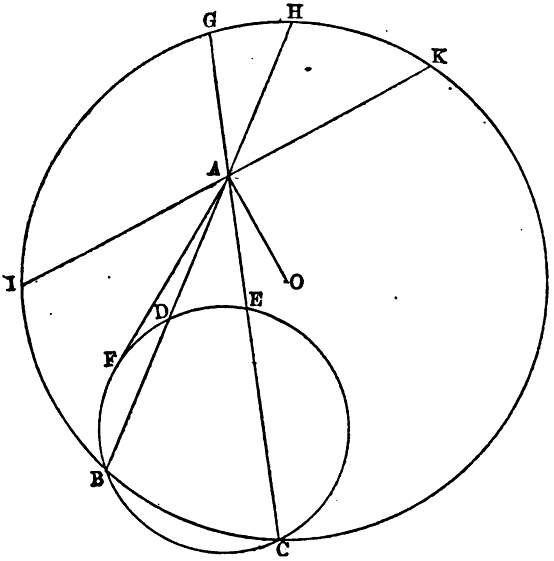
THEOREM 50.

If two secants be drawn to a circumference from a point without it, they are reciprocally proportional to their outer parts.

Let there be two secants A B and A C drawn to the circumference B C D from a point A without it, and let A D and A E be their outer parts.

Let there be a tangent A F drawn from the point A to the circumference B C D; then A F is a mean proportional to A C and its outer part A E [45]. Let C A be produced to a distance A G equal to A E; let there be a circumference O through the points B, C and G, and let B A be produced to a point H thereof; then A B is to A C as A G is to A H [47]: but A G is equal to A E [*Const.*]; therefore A B is to A C as A E is to A H.

Let the point A be joined to the centre O and, through the same point, let I K be a chord perpendicular to A O; then I K is bisected



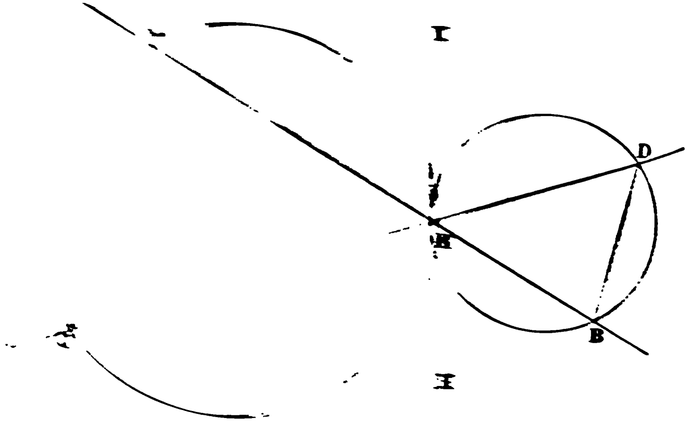
in the point A [I. 44 i]. Because A I is a mean proportional to A C and A G [50], and because A E is equal to A G [*Const.*], the line A I is also a mean proportional to A C and A E; therefore A I is equal to A F [xxii]. Because A H is to A I as A I is to A B [49] and because A F is equal to A I, the line A H is to A F as A F is to A B : but A D is also to A F as A F is to A B [45]; then A H is equal to A D [xvii]. Now, A B is to A C as A E is to A H [*Dem.*]; therefore A B is to A C as A E is to A D, which W. T. B. D.

COROLLARY. *If the distances from the vertex of an angle to the intersections of the sides with two straight lines be reciprocally proportional, the four intersections are points of the same circumference* (48).

[illegible]

If two circumferences be tangent to one another, and if [illegible] contingence, the inter- [illegible] circumferences are pro- [illegible]

Let [illegible] circumferences tangent in E, and [illegible] circumferences through the [illegible]



Let A C and B D be joined, and let I H be a tangent to the two circumferences at the point E [illegible]. Because the angle A C E is equal to the angle A E I [illegible], and the angle B D E equal to the angle B E H; because also the angle A E I is equal to its vertical B E H [illegible], the alternate angles A C E and B D E are equal to each other [illegible] and the lines A C and D B are parallel to one another [21 i]: therefore A E is to B E as C E is to D E [38 i], which W. T. B. D.

Corollary I. *If an angle or vertical angles, be subtended by parallels, the circumference passing through the vertex and the extremities of each of the subtending lines, is tangent to that passing through* [illegible] *and the extremities of another.*

COROLLARY II. *The intercepts of a common secant through the point tangence of two tangent circumferences, are proportional to their dii.*

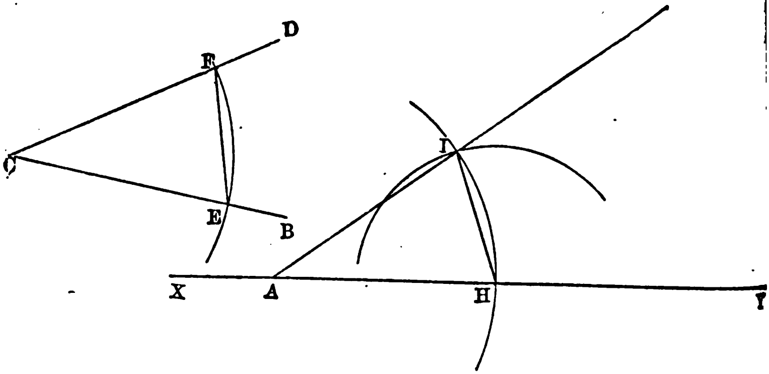
Scholium. The intercepts of a common secant through the point tangence of two equal tangent circumferences, are equal to each ther (I. 72).

PROBLEMS.

PROBLEM 1.

At a point given in a given straight line, to construct an angle equal to a given angle.

Let X Y be the given straight line, A the point given in it and B C D the given angle.



From the centres C and A, with equal radii, describe two circumferences cutting C B, C D and A Y in E, F and H; join E F; from the centre H, with a radius equal to E F, describe a circumference cutting the circumference A in a point I, and join A I ∴ the angle H A I is the required angle.

Join H I; then H I is a chord of the circumference A. Because the circumferences C and A have equal radii [*Const.*], they are equal to each other [I. 38]. Because H I is a radius of the circumference H, the line H I is equal to E F [*Const.*]. But H I and E F are chords of the equal circumferences C and A; then the arcs H I and E F are equal to each other [I. 43]: therefore the centre angles H A I and E C F are equal to each other [26]; that is, the angle H A I is equal to the angle B C D, which was to be done.

Problem 2. *To find the sum of two, or any number of, given angles.*

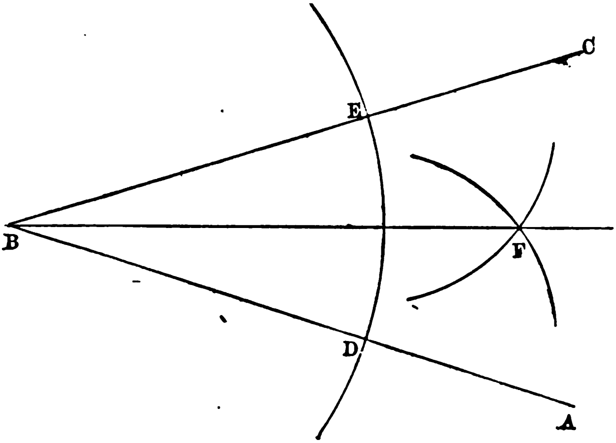
Problem 3. *To find the product of a given angle by a given number.*

Problem 4. *To find the difference of two given angles.*

Problem 5.

To bisect a given angle.

Let A B C be the given angle.



From the centre B, describe a circumference cutting the sides of the angle A B C in the points D and E; from the centres D and E, with a radius greater than half the distance of the points D and E, describe two circumferences secant in F, and draw a straight

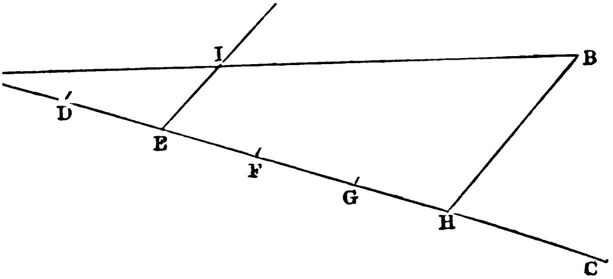
use B M and H N are parallel to each other [*Const.*], A B is to A M is to M N [38]; but B H and A M are respectively equal and E F: therefore A B is to C D as E F is to M N; that is, a fourth proportional to the lines A B, C D and E F, which B. D.

ILEM 7. *To find a third proportional to two given straight lines.*

PROBLEM 8.

livide a given straight line in a given ratio.

A B be the given straight line to be divided in the ratio of *three.*



ugh the point A, draw a straight line A C in any direction; y distance A D, and set it off upon A C *five* times successively D E, E F, F G and G H; join H B, and through the point E, I parallel to H B [I. *Prob.* 10]: the line A B is divided by at I, in the ratio of *two* to *three.*

use A D is a common measure to A E and E H, and because d E H contain that measure *two* and *three* times respectively, to E H as *two* is to *three* [x]. Because E I and H B are to each other [*Const.*], A I is to I B as A E is to E H [38]; re A I is to I B as *two* is to *three* [i]: that is, A B is divided in io of *two* to *three*, W. W. T. B. D.

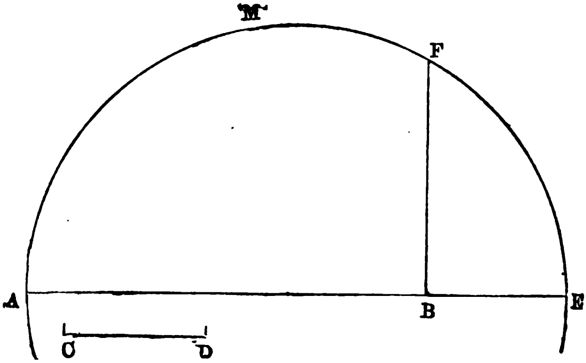
Problem 9. *To divide a given straight line proportionally to several given numbers.*

Problem 10. *To divide a given straight line similarly to a given divided straight line.*

Problem 11.

To find a mean proportional to two given straight lines.

Let A B and C D be the two given straight lines.



Produce A B to a distance B E equal to the length of C D, and on A E describe a circumference A M E [I. *Def.* 26]; through the point B draw a perpendicular to A E, meeting the circumference A M E in a point F: the line B F is the required mean proportional.

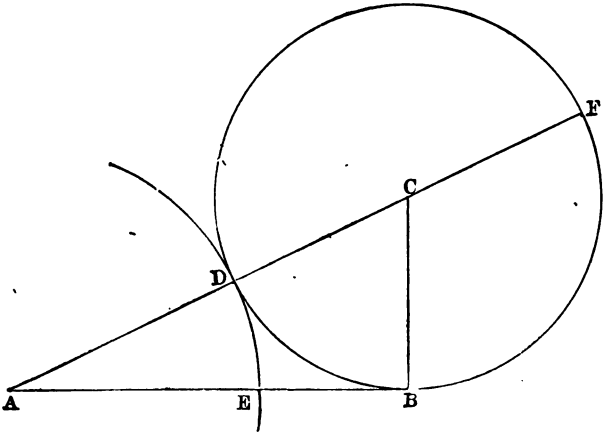
Because A E is a diameter of the circumference A M E [*Const.*], and B F is a perpendicular to A E [*Const.*], the line B F is a mean proportional to A B and B E [49 ii]: but C D is equal to B E [*Const.*]; therefore B F is a mean proportional to A B and C D, which W. T. B. D.

Problem 12.

To divide a given straight line into extreme and mean ratio.

Let A B be the given straight line.

Through the point B draw a perpendicular B C equal to half of



A B [I. *Prob.* 5] and join A C; from the centre C describe, through the point B, a circumference cutting A C in a point D, and from the centre A describe, through the point D, a circumference cutting A B in a point E: the line A B is divided by the point E into extreme and mean ratio.

Produce A C to a point F of the circumference C; then D F is a diameter of the circumference C. Because A B is perpendicular to the radius B C [*Const.*], the line A B is a tangent to the circumference C [I. 54]; then A F is to A B as A B is to A D [46]: therefore the difference of A F and A B is to A B as the difference of A B and A D is to A D [xv]. Because the radius of the circumference C is equal to half of A B [*Const.*], the diameter D F is equal to A B [vi]; also the radii A D and A E of the circumference A are equal to each other; then the difference of A F and D F is to A B as the difference of A B and A E is to A E, or A E is to A B as E B is to A E: therefore A B is to A E as A E is to E B; that is, A B is medially divided in the point E [*Def.* 22], which W. T. B. D.

BOOK III.

ON PLANE FIGURES.

BOOK III.

ON PLANE FIGURES.

CHAPTER I.

ON RECTILINEAR PLANE FIGURES.

SECTION I.

ON POLYGONS GENERALLY.

DEFINITIONS.

1. A *plane figure* is a determinate portion of a plane.

It has been shown [*Gen. Def.* 5] that every portion of a surface is limited by lines; consequently, a plane figure is limited by one or more lines. A plane figure is *rectilinear* or *curvilinear*, according to the kind of lines by which it is limited. Whence

2. A *rectilinear* plane figure, or *polygon*, is limited by straight lines, called its *sides;* two sides having one point in common are *adjacent* to each other. The *perimeter* of a polygon is the sum of all its sides; the *semiperimeter* is half that sum.

It is evident that no plane figure can be formed by less than three straight lines.

3. The angles formed by adjacent sides of a polygon are

the *interior* angles, or simply the *angles*, of the polygon; those formed by each side with the prolongation of its adjacent, are the *exterior* angles; their vertices are the *angular* points, or *vertices* of the polygon. *Adjacent* vertices of a polygon are the extremities of the same side.

4. A *diagonal* line, or simply *diagonal*, joins two vertices not adjacent to each other; the diagonals *adjacent* to one side are terminating at its extremities. The *primary* diagonals subtend the interior angles; the *secondary* diagonals subtend the angles formed by the primary diagonals and the sides.

5. A *convex* polygon has only projecting angles; a *concave* polygon has one or more receding angles.[1]

Particular names have been given to polygons of a certain number of sides. Thus

6. A polygon of	three	sides is a	*triangle*, or *trigon;*
,,	four	,,	*quadrangle*, or *tetragon;*
,,	five	,,	*pentagon;*
,,	six	,,	*hexagon;*
,,	seven	,,	*heptagon;*
,,	eight	,,	*octagon;*
,,	nine	,,	*enneagon;*
,,	ten	,,	*decagon;*
,,	twelve	,,	*dodecagon;*
,,	fifteen	,,	*pentedecagon.*

---

[1] Whenever in The Elements a polygon is spoken of, without anything further being said, a *convex* polygon is always meant.

7. An *equilateral* polygon is that of which all the sides are ual to each other; an *equiangular* polygon is that of which the angles are equal to each other.

THEOREM 1.

In every polygon there are as many vertices as there are es.

For if the sides be numbered, the first will begin at the first tex and meet the second side; the second will begin at the second tex and meet the third side; and so on to the last side, which will gin at the last vertex and meet the first side at the first vertex.

COROLLARY. *The straight lines joining a point within a polygon to its angular points, divide the polygon into as many triangles as re are sides.*

THEOREM 2.

If all the diagonals be drawn from one vertex of a polygon, re are three diagonals less than there are sides.

Each vertex may be joined to all the others by as many straight es as there are other vertices [*Post.* V], that is, one less than there sides: but two of these straight lines are sides of the polygon, d the others, diagonals; therefore there are three diagonals less n there are sides, W. W. T. B. D.

COROLLARY. *The diagonals drawn from one vertex, divide the polygon o as many triangles as there are sides but two.*

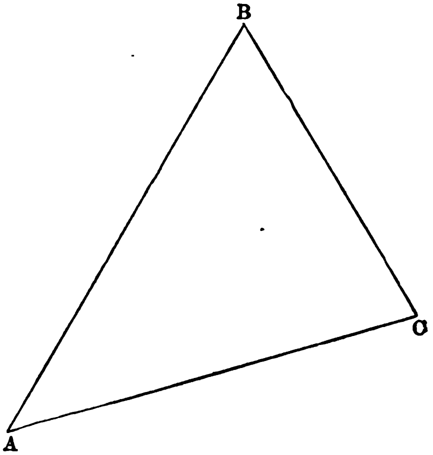
THEOREM 3.

(*Eucl.* I. 32.)

The three angles of every triangle are together equal to two ht angles.

Let A B C be a triangle.

Because the straight lines A B and B C meet in B, the transversal A C forms two interior angles B A C and B C A, which are together supplementary to the angle A B C [II. 21]: therefore the three angles are together equal to two right angles, W. W. T. B. D.



COROLLARY I. *An exterior angle of a triangle, is equal to the sum of the two opposite interior angles.*

COROLLARY II. *If two angles of a triangle be equal to two angles of another, the third angle of the one is equal to the third of the other.*

COROLLARY III. *In a triangle there cannot be more than one right, or obtuse, angle.*

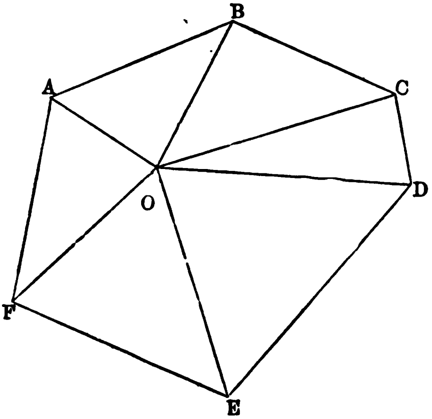
Scholium. If two angles of a triangle be known, the third is determined.

THEOREM 4.

(*Eucl.* I. 32 i.)

All the interior angles of a polygon are together equal to twice as many right angles as the figure has sides minus four right angles.

Let A B C D E F be a convex polygon. Let a point O within the polygon be joined to each vertex: the polygon is now divided into as many triangles as there are sides. Now, the sum of the angles of hese triangles is the sum of the interior angles of the polygon, toether with the angles formed at O; but the angles of each triangle re together equal to wo right angles [3], nd the angles formed t O are together equal o four angles [II. 12]: therefore the anles of the polygon re together equal to wice as many right ngles, as the figure as sides minus four ight angles, W. W. . B. D.



Corollary. *The exerior angles of a convex olygon, are together qual to twice four right angles.*

Scholium. The four angles of a quadrangle are together equal to our right angles.

SECTION II.

ON TRIANGLES.

DEFINITIONS.

8. The interior, or exterior, angle formed by two sides of triangle is *opposite* to the third side, and to the angles ormed by these lines.

9. A *right-angled* triangle is that which has a right angle; he right angle is formed by the *right* sides, and is ubtended by the *hypothenuse;* the angles *at the hypothe-*

side are formed by the hypothenuse. An *obtuse-angled* triangle has an obtuse angle; an *acute-angled* triangle has three acute angles.

It is evident that the angles at the hypothenuse of any right-angled triangle are complementary, and that any triangle must be either right-angled, obtuse-angled or acute-angled.

10. An *isosceles* triangle has two sides equal to each other; the third side, or *base*, forms the *angles at the base*; the opposite angular point is the *apex* of the triangle and is the vertex of the angle *at the apex*. A *scalene* triangle has all its sides unequal.

Hence, the base of an isosceles triangle subtends symmetrically the angle at the apex.

11. The *altitude* of any vertex of a triangle is the perpendicular drawn from it upon the opposite side; the vertex is the *apex*, and the opposite side is the *base* of that altitude. The altitudes of the three vertices of a triangle are the *altitudes* of the triangle.[1]

12. The *median* to any side of a triangle is the straight line passing through its middle point and through the opposite vertex; the point of intersection of the medians is called the *centre* of the triangle.[2]

THEOREM 5.

(*Eucl.* I. 20.)

In any triangle, one side is less than the sum of the two others and greater than their difference.

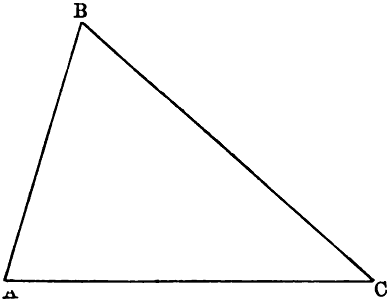
---

[1] In The Elements, by *the altitude* of a triangle, it is always meant the altitude upon the side which is the lowest in the diagram, or, if there is no diagram, any one of the three sides, unless the sense indicates otherwise.

[2] In Mechanics, this point is called the *centre of gravity* of the triangle.

Let A B C be any triangle.

irstly, each of the straight lines A B, B C and A C which join the ints A, B and C is less than the sum of the two others [I. 6 i]; erefore any one side of the angle A B is less than e sum of the two others.



Secondly, because the side C is less than the sides B and B C together, the dif- rence between the side A C d the side B C is less than e difference between the des AB and B C together d the side B C [iii], that is, ss than A B: therefore the side A B is greater than the difference of e other sides. The same reasoning would prove that each of the des B C and A C is greater than the difference of the other sides: ence any side of the triangle is greater than the difference of her sides.

Therefore, any side of the triangle A B C is less than the sum the two others, and greater than their difference, W. W. T. B. D.

versely

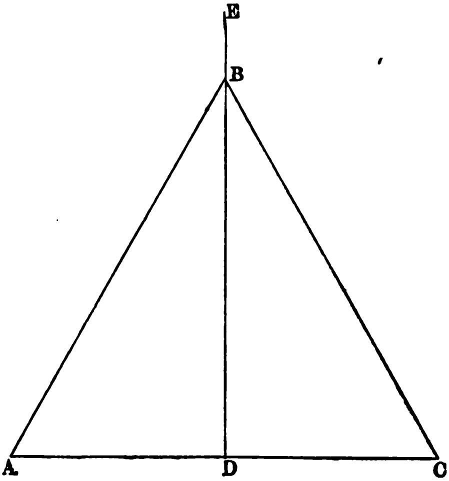
If one of three straight lines be not less than the sum of the others, not greater than their difference, they cannot form a triangle.

COROLLARY. *If there be two triangles having two vertices of the one inciding with two vertices of the other, and if the third vertex of the e be without the other triangle, so that two sides intersect each other, e sum of the intersecting sides is greater than that of the other sides at are not common to both triangles.*

THEOREM 6.

If two angles of a triangle be equal to each other, the iddle perpendicular on the side forming them, passes rough the third vertex.

Let the angles A and C of the triangle A B C be equal to each other, and let D E be the middle perpendicular on the side A C.



Let the triangle be folded upon D E: then A D will coincide with D C, and the point A with its symmetric C. Because the angles A and C are equal to each other, the sides A B and C B will coincide: therefore A B and C B will meet the line of symmetry D E in the same point [I. 30]; that is, the perpendicular D E will pass through the vertex B, which W. T. B. D.

Inversely

If the middle perpendicular on any side of a triangle does not pass through the opposite vertex, the two angles formed by that side are not equal to each other.

Corollary. *The middle perpendicular on every side of an equiangular triangle, passes through the opposite vertex.*

Theorem 7.

(*Eucl.* I. 6.)

If two angles of a triangle be equal to each other, the sides opposite to them are equal to each other.

Let the angles A and C of the triangle A B C be equal to each other.

Let D E be the middle perpendicular on A C. Because D E passes

e vertex B [6],
oints A and
metric to D E
lines A B and
symmetric to
eefore A B is
B C, which
. Inversely
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al sides of a tri-
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RY I. *An equi-*
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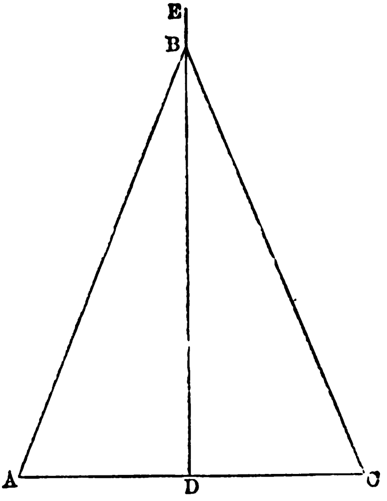

OREM 8.

ucl. I. 5.)

gles at the base
sceles triangle,
to one another.

sides A B and
triangle A B C
to each other.
be the bisectrix
le A B C; then
ces A and C,
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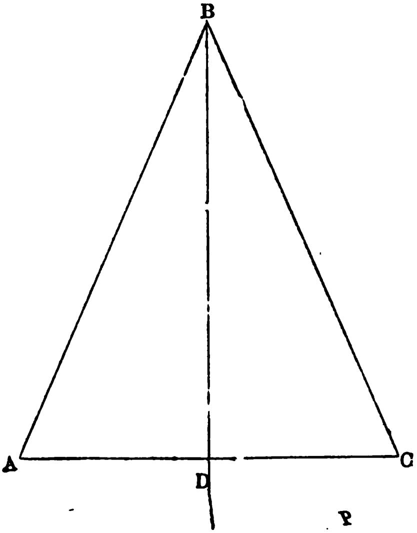


symmetric to B D [II. 6 ii], and the two portions A D and D C of the base, are symmetric to the same line [I. 27 ii], the angles B A D and B C D have their sides symmetric to B D: therefore they are equal to each other [II. 19], W. W. T. B. D. Inversely

The sides opposite to unequal angles of a triangle, are unequal.

Corollary I. *All the interior, or exterior, angles of an equilateral triangle, are equal to one another.*

Corollary II. *If the equal sides of an isosceles triangle be produced beyond the base, they form with the base two equal exterior angles.*

Corollary III. *If the angles of a triangle be unequal, the triangle is scalene.*

Corollary IV. *The altitude of an isosceles triangle is the bisectrix of the angle at the apex,* and conversely.

Corollary V. *The altitude of an isosceles triangle divides its perimeter into two symmetric parts.*

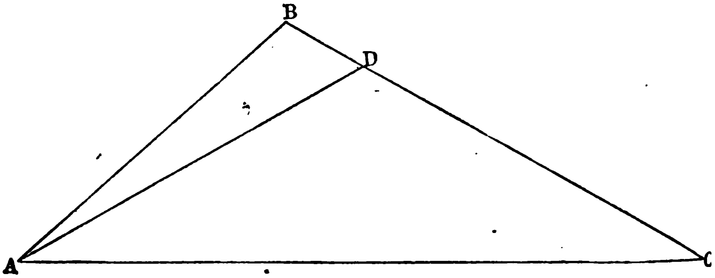
Scholium. The equal sides of an isosceles right-angled triangle are those which form the right angle (I. 11).

Theorem 9.

(*Eucl.* I. 19.)

A greater angle of a triangle is opposite a greater side.

In the triangle A B C, let the angle B A C be greater than the angle B C A.



Let an angle D A C be equal to the angle B C A: then the side

A, in the triangle A D C, is equal to the side D C [7]. Because A is less than B D and D A together [5], B A is less than B D and C together [*Post.* II]; that is, B A is less than B C, which W.T.B.D. versely

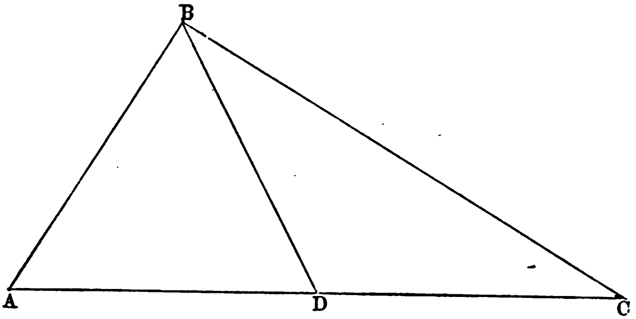
A greater side of a triangle is opposite a greater angle.

Scholium. The greatest angle of a triangle is opposite the greatest le, and the smallest angle opposite the smallest side, and conversely.

THEOREM 10.

In a right-angled triangle, the median from the vertex of e right angle is equal to half the hypothenuse.

Let A B C be a triangle right-angled in B.



Let the angle A B D be equal to the angle A: then the triangle B D is isosceles [7]; that is, B D is equal to A D. Because the angles B C and C are complementary to equal angles [3], they are equal to ch other [II. 15 i]; consequently the triangle D B C is isosceles [7], d B D is equal to D C: therefore A D is equal to D C [i]; that is, D is a median of the triangle A B C, and is equal to half of A C, hich W. T. B. D. Inversely

An angle of a triangle is not a right angle, if the median from its rtex be not equal to half the opposite side.

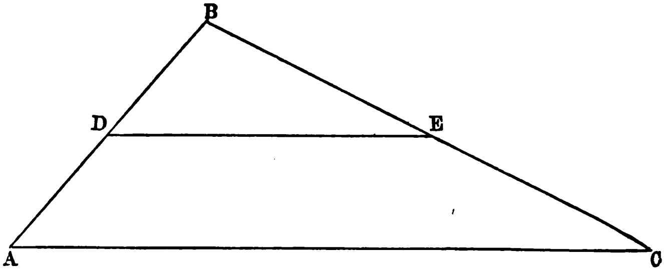
COROLLARY I. Conversely, *an angle of a triangle is a right angle, if the median through its vertex be equal to half the opposite side.*

COROLLARY II. *In any triangle, the median through an obtuse angle is less than the opposite side, and the median through an acute angle is greater than the opposite side,* and conversely.

THEOREM 11.

The middle line of two sides of a triangle, is equal to half the third side.

Let D E be the middle line of the sides A B and B C of the triangle A B C.



Because the sides of the angle A B C are cut proportionally by D E and A C, the lines D E and A C are parallel to each other [II. 39] : therefore D E is equal to half of A C [II. 41 i], which W. T. B. D.

COROLLARY I. *The middle line of any two sides of a triangle is parallel to the third side.*

COROLLARY II. *The median to each side of a triangle bisects the middle line of the other sides.*

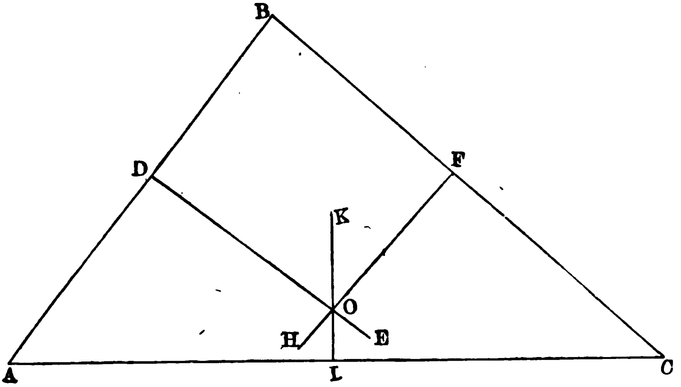
COROLLARY III. *The middle lines of the sides of an isosceles or equilateral triangle, divide it into four isosceles or equilateral triangles.*

COROLLARY IV. *The parallels through the vertices of an isosceles or equilateral triangle, form an isosceles or equilateral triangle.*

THEOREM 12.

In every triangle, the middle perpendiculars on the three sides meet at the same point.

Let D E, F H and I K be the middle perpendiculars on the sides of the triangle A B C.



Because A B and B C are not parallel, the lines D E and F H perpendicular thereon will meet, if indefinitely produced [I. 18 ii]. Let O be the intersection of D E and F H. Because the point O belongs to the middle perpendicular on A B, it is equally distant from the vertices A and B [I. 14]; because the same point belongs to the middle perpendicular on B C, it is also equally distant from the vertices B and C; hence the point O is equally distant from the vertices A and C [i], and belongs to the middle perpendicular on A C [I. 15 i]: therefore I K will meet D E and F H at O, which W. T. B. D.

COROLLARY. *The three altitudes of a triangle meet at the same point.*

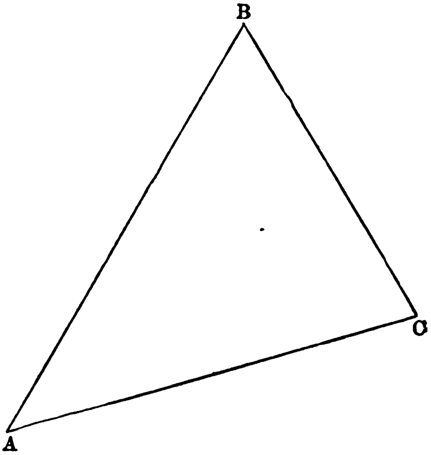
Scholium. The middle perpendiculars on the sides of a triangle are the altitudes of the triangle formed by the middle lines of the sides.

THEOREM 13.

In every triangle, the bisectrices of the three angles meet at the same point.

Let A B C be a triangle.

Because the straight lines A B and B C meet in B, the transversal A C forms two interior angles B A C and B C A, which are together supplementary to the angle A B C [II. 21]: therefore the three angles are together equal to two right angles, W. W. T. B. D.



COROLLARY I. *An exterior angle of a triangle, is equal to the sum of the two opposite interior angles.*

COROLLARY II. *If two angles of a triangle be equal to two angles of another, the third angle of the one is equal to the third of the other.*

COROLLARY III. *In a triangle there cannot be more than one right, or obtuse, angle.*

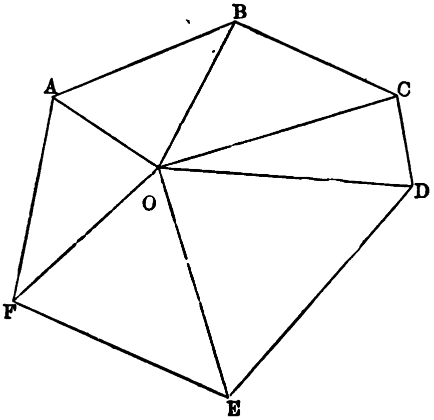
Scholium. If two angles of a triangle be known, the third is determined.

THEOREM 4.

(*Eucl.* I. 32 i.)

All the interior angles of a polygon are together equal to twice as many right angles as the figure has sides minus four right angles.

Let A B C D E F be a convex polygon. Let a point O within the polygon be joined to each vertex: the polygon is now divided into as many triangles as there are sides. Now, the sum of the angles of these triangles is the sum of the interior angles of the polygon, together with the angles formed at O; but the angles of each triangle are together equal to two right angles [3], and the angles formed at O are together equal to four angles [II. 12 ii]: therefore the angles of the polygon are together equal to twice as many right angles, as the figure has sides minus four right angles, W. W. T. B. D.



COROLLARY. *The exterior angles of a convex polygon, are together equal to twice four right angles.*

Scholium. The four angles of a quadrangle are together equal to four right angles.

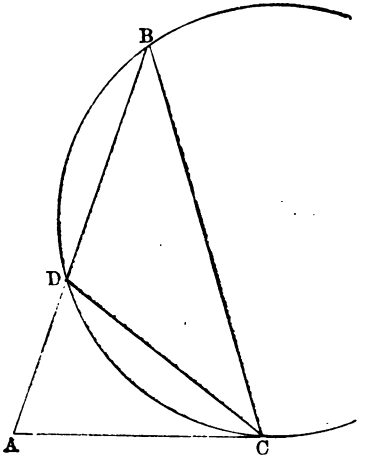
SECTION II.

ON TRIANGLES.

DEFINITIONS.

8. The interior, or exterior, angle formed by two sides of a triangle is *opposite* to the third side, and to the angles formed by these lines.

9. A *right-angled* triangle is that which has a right angle; the right angle is formed by the *right* sides, and is subtended by the *hypothenuse;* the angles *at the hypothe-*



Let C D be the bisectrix of the angle A C B and let there be a circumference BCD through the points B, C and D. Because the angle ABC is equal to half the angle ACB [*Hyp.*] the angle A B C is equal to the angle B C D; therefore C D is equal to B D [7]: also the angle ABC is equal to the angle A C D; therefore the angle A D C is equal to the angle A C B [3 ii]. Because the angles B A C and A C B are equal to each other [8], the angle B A C is equal to the angle A D C [i]; consequently C D is equal to A C [7]: but C D is equal to B D [*Dem.*]; therefore A C is equal to B D [i]. Because the angle A C D and the inscribed angle D B C are equal to each other, A C is a tangent to the circle B C D (II. 29 ii]; thence A B is to A C as A C is to A D: therefore A B is to B D as B D is to A D; that is, the point D divides A B in extreme and mean ratio, and the base A C is equal to the mean B D, which W. T. B. D.

Corollary I. Conversely, *if the base of an isosceles triangle be equal to the mean of one of the equal sides, each angle at the base is double of that at the apex.*

Corollary II. *If each angle at the base of an isosceles triangle be double of that at the apex, the bisectrix of each of them is equal to the base and divides the opposite side in extreme and mean ratio.*

Corollary III. *If the angle at the apex of an isosceles triangle be equal to three times each angle at the base, each of the equal sides is equal to the mean of the base,* and conversely.

SECTION III.

ON THE COMPARISON OF TRIANGLES.

DEFINITIONS.

uality of triangles is best proved by the method of superposition, nsists in applying them to one another, either by folding the plane tains both, or by slipping one over the other, in order to ascertain hey will coincide or not.

ws immediately from the definition of equality [*Gen. Def.* 18] that gles which are equal to one another, that is, which coincide exactly lied to one another, have all their parts, sides and angles, equal, ch.

he *homologous* sides, angles or vertices, of two equal s are those which coincide when the two triangles le to coincide.

a triangle a greater side is opposite a greater angle, so it must be in equal triangles; that is, the homologous sides are opposite logous angles. It is evident that, if all the parts of two triangles, l angles, be equal, each to each, so that the sides opposite the les be equal to each other, the two triangles will be equal, for, if pplied to each other, they will coincide in all their parts.

he definitions of similar figures [*Gen. Def.* 19], it follows that the of two triangles is best proved by showing that they will be equal her if they be reduced to the same magnitude.

he *homologous* sides, angles, or vertices, of two triangles are those which coincide, when the two s are reduced to the same magnitude and made cide.

reduction of a figure, every part of it preserves the same form , and similar angles are equal to each other [II. 1]. Thence,

POSTULATE X.

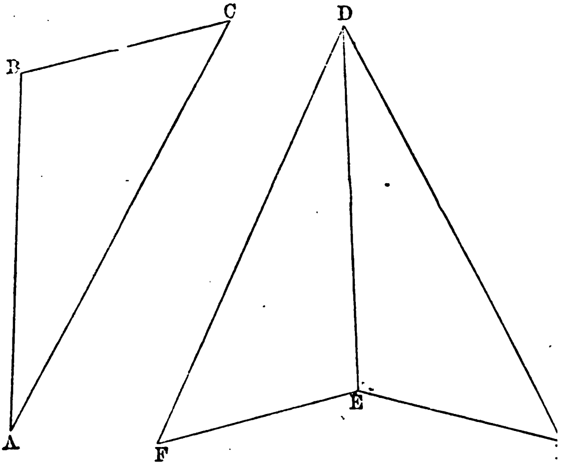
eduction of a plane figure does not alter its angles.

THEOREM 16.

(*Eucl.* I. 26.)

If two triangles have one side of the one equal to one s the other, and if the two angles formed by the equal side one be equal to the two angles formed by the equal side other, each to each, the two triangles are equal to one a

Let A B C and D E F be two triangles in which the side equal to the side D E, the angle A equal to the angle D, angle B equal to the angle E.



Let the triangle A B C be placed in a position D H E on of D E opposite to that of the triangle D E F, so that A B with its equal D E, and the angle A be adjacent to its equal the angle B will be adjacent to its equal E. Thus the ang is equal to the angle F D E, and the angle H E D is equ angle D E F; that is, the line D E is the bisectrix of both t H D F and H E F: therefore the points F and H are sym D E [II. 9 ii].

Let the figure be folded upon D E; then the symmetric points F d H will coincide with each other; also E H will coincide with F, and D H with D F: that is, the triangle D H E will coincide th the triangle D F E: therefore the triangles D E F and D H E e equal to each other. But the triangle D H E is no other than e triangle A B C [*Const.*]; therefore the triangles ABC and DEF e equal to each other, W. W. T. B. D.

Corollary I. *If two triangles have two angles of the one equal, each each, to two angles of the other, and if the sides subtending one of the ual angles, be also equal, the triangles are equal to each other.* (Eucl. 26.)

Corollary II. *If two right-angled triangles have the hypothenuse nd an angle at the hypothenuse of the one, respectively equal to the ypothenuse and an angle at the hypothenuse of the other, they are ual to each other.*

Corollary III. *Two right-angled triangles are equal to each other they have one of the right sides equal, and either the opposite angle, the angle formed by the equal sides and the hypothenuse, also equal.*

Corollary IV. *If two isosceles triangles have the angle at the apex, an angle at the base, and the opposite side of the one, respectively ual to the angle at the apex, or to an angle at the base, and the opposite le of the other, they are equal to each other.*

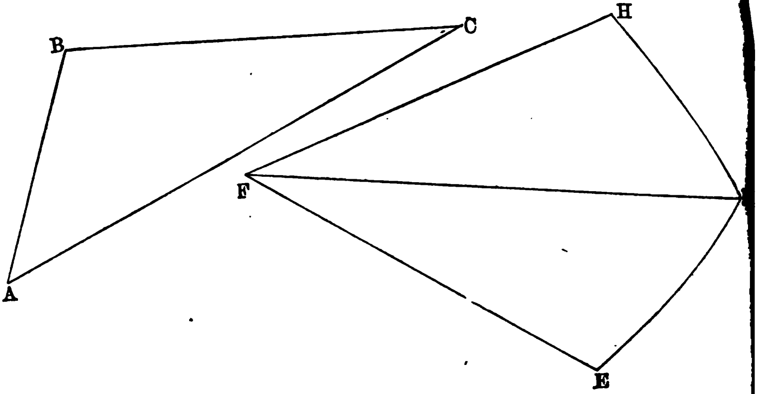
Theorem 17.

(*Eucl.* I. 4.)

If two triangles have two sides of the one equal to two les of the other, each to each, and if the angles formed by e equal sides be equal, the triangles are equal to each other.

Let two triangles A B C and D E F have the side A B equal to e side D E, the side A C equal to the side D F, and the angle A ual to the angle D.

Let the triangle A B C be placed in a position D H F on the side of F, opposite to that of the triangle D E F, so that A C coincide th its equal D F, and A B be adjacent to its equal D E; then the angle A will be adjacent to its equal D. Thus D H is equal to D E, and the angle H D F is equal to the angle E D F.



Let the figure D E F H be folded on D F; then the side D E will coincide with the side D H, and the point E with the point H; also the side E F will coincide with the side H F [I. 3]; that is, the triangle D E F will coincide with the triangle D H F; therefore the triangles D E F and D H F are equal to each other. But the triangle D H F is no other than the triangle A B C [*Const.*]; therefore the triangles A B C and D E F are equal to each other, W. W. T. B. D. Inversely

If two triangles have two sides of the one equal to two sides of the other, each to each, and if the third sides be unequal to each other, the angles formed by the equal sides are unequal to each other.

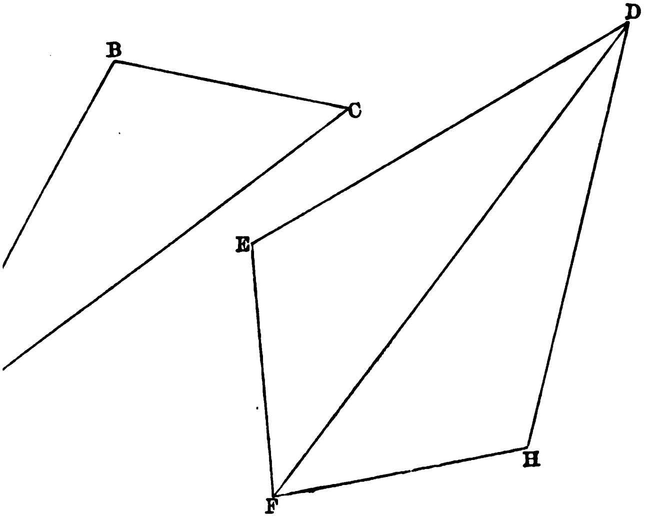
COROLLARY. *If two right-angled triangles have the perpendicular sides of the one equal to the perpendicular sides of the other, each to each, they are equal to each other.*

THEOREM 18.

(*Eucl.* I. 8.)

If two triangles have the three sides of the one equal to the three sides of the other, each to each, they are equal to each other.

.B C and D E F be two triangles having the sides A B, B C
severally equal to the sides D E, E F and D F.



he triangle A B C be placed in a position D H F on the side
pposite to that of the triangle, D E F, so that A C coincide
equal D F, and B C be adjacent to its equal E F. Thus D H
l to D E, and forms with it the angle H D E; also H F is
E F, and forms with it the angle E F H. Because the points E
ire symmetric points of both the angles E D H and E F H
the line F D is the bisectrix of the angles E D H and E F H
; therefore the points E and H are symmetric to F D [II. 6].
he figure D E F H be folded upon F D; then the sym-
points E and H will coincide with each other; also H D
icide with E D, and H F with E F; that is, the triangle D H F
icide with the triangle D E F; therefore the triangles D E F
I F are equal to each other. But the triangle D H F is no
ian the triangle A B C [*Hyp.*]; therefore the triangles A B C
F are equal to each other, W. W. T. B. D. Inversely

If two triangles have two sides of the one equal to two sides of the other, each to each, and if the angles formed by these sides be not equal to each other, the third sides are unequal.

COROLLARY I. *Two equilateral triangles which have one side of the one equal to one side of the other, are equal to each other.*

COROLLARY II. *Two isosceles triangles which have the base and side of the one equal to the base and side of the other, each to each, are equal to each other.*

COROLLARY III. *The middle lines of the sides of a triangle divide the triangle into four equal triangles.*

THEOREM 19.

(*Eucl.* I. 24.)

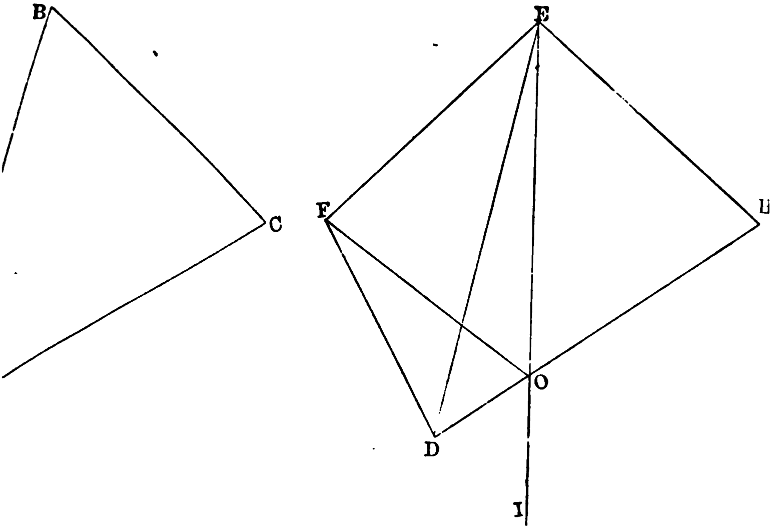
If two triangles have two sides of the one equal to two sides of the other, each to each, and if the angles formed by the equal sides be not equal to each other, the third side subtending the greater angle is greater than the third side subtending the lesser.

Let A B C and D E F be two triangles having the side A B equal to the side D E, and the side B C equal to the side E F; and let the angle B be greater than the angle E.

Let the triangle A B C be placed in a position D E H on the side of D E opposite to that of the triangle D E F, so that A B coincide with its equal D E, and B C be adjacent to its equal E F; then the angle D E H is greater than the angle D E F, and E H is equal to E F; that is, H and F are symmetric points of the angle H E F.

Let E I be the bisectrix of the angle F E H. Because the angle D E H is greater than the angle D E F, the bisectrix E I of the whole angle F E H will fall within the angular space of the angle D E H, and cut D H in a point O. Let O F be joined. Because the point O belongs to the bisectrix of the angle F E H, the line O F is equal to O H; but O F and O D are together greater than D F

]: therefore H O and O D are together greater than D F; that H D is greater than D F. But the triangle H E D is no other



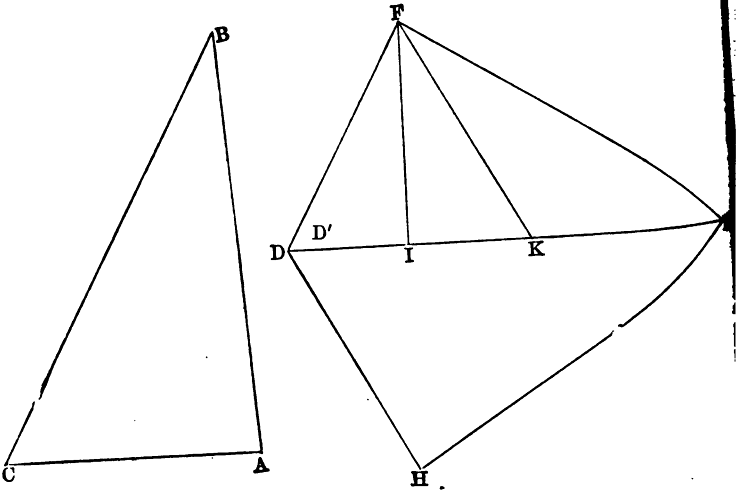
an the triangle A B C, and the side H D no other than the side . C [*Const.*]; therefore the side A C is greater than the side D F, hich W. T. B. D. Inversely

If two triangles have two sides of the one equal to two sides of the her, each to each, and the third sides be not equal to each other, the eater third side subtends a greater angle. (Eucl. I. 25.)

THEOREM 20.

If two triangles have one angle of the one equal to one ngle of the other, if the sides forming another angle be equal, ch to each, and if the third angle be acute in both triangles, e two triangles are equal to each other.

Let A B C and D E F be two triangles in which the angle B is equal to the angle E, the side A B equal to the side E F, the side A C equal to the side D F, and let the angles C and D be both acute.



Let the triangle A B C be placed in a position E H D′ on the side of E D opposite to that of the triangle E D F, so that the points B and E coincide with each other, and B C coincide with E D, and let D′ be the point of E D coinciding with the point C; then the side A B will be adjacent to its equal E F. Thus E H is equal to E F, or the points H and F are symmetric points of the angle H E F; also the angle H E D is equal to the angle F E D, or E D is the bisectrix of the angle H E F: therefore the points H and F are symmetric to E D [II. 6].

Let F I be a perpendicular to E D drawn from the point F, and F K an oblique equal to F D drawn from the same point; now let the figure E H F D be folded upon E D: then the point H will coincide with its symmetric F. Because H D′ is equal to F D [*Hyp.*], and because the point F cannot be joined to E D by more than equal straight lines [II. 13 ii], the line H D′ will coincide with F D ith F K; that is, the point D′ will coincide with the point D or the point K. Because F D and F K subtend right angles, the angle E is an acute angle, and F K E is an obtuse angle [II. 22 ii]; but angle H D′ E is acute [*Hyp.*], then H D′ coincides with F D and point D′ with D: that is, the triangle H D′ E will coincide with F; therefore they are equal to each other. But the triangle ′E is no other than the triangle A B C [*Const.*]; therefore the ngles A B C and D E F are equal to each other, W. W. T. B. D.

orollary I. *If two triangles have one angle of the one equal to of the other, if the sides forming another angle be equal, each to , and if the third angle be obtuse in both triangles, the two ngles are equal to each other.*

orollary II. *If two triangles have one angle of the one equal to of the other, and if the sides forming another angle be equal, each ach, the third angle of each triangle is either equal or supple- tary to the third angle of the other.*

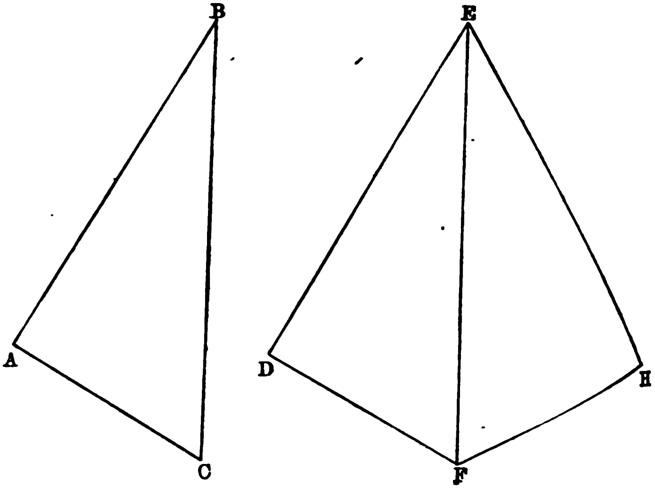
Theorem 21.

f two right-angled triangles have the hypothenuse and one he right sides of the one, respectively equal to the hypo- use and one of the right sides of the other, the two triangles equal to each other.

t A B C and D E F be two right-angled triangles having the thenuse B C equal to the hypothenuse E F, and the side A C l to the side D F.

t the triangle A B C be placed in a position E H F on the side F opposite to that of the triangle D E F, so that B C coincide its equal E F and A C be adjacent to its equal D F; then H F is l to D F. Because the point F is equally distant from the sides e angle D E H [I. 11], the line E F is the bisectrix of the angle F [II. 11 iii]; then the angle D E F is equal to the angle H E F: efore the triangles E H F and E D F are equal to each other

[16 ii]. But the triangle E H F is no other than the triangle A B C



[*Const.*]; therefore the triangles A B C and D E F are equal to each other, W. W. T. B. D.

COROLLARY. *If two isosceles right-angled triangles have the hypothenuse of the one equal to the hypothenuse of the other, they are equal to each other.*

THEOREM 22.

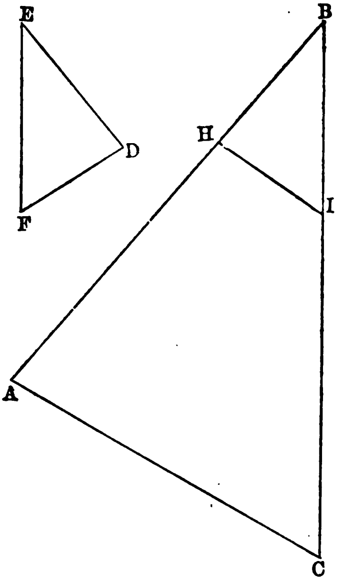
Two similar triangles have their homologous angles equal, each to each, and their homologous sides proportional.

Let the triangles A B C and D E F be similar to each other, and let the vertices D, E and F be respectively homologous to the vertices A, B and C.

If these two triangles be equivalent [*Gen. Def.* 21], they are equal to each other [*Gen. Def.* 18]: therefore their homologous angles are equal, each to each; also their homologous sides are proportional, having to one another the ratio of equality.

iangles A B C and D E F
uivalent, and if they be
o the same magnitude
22], they will then be
each other, and their
s angles will be equal,
ach : but the reduction
iangles to the same
will not have altered
les [*Post.* X]; therefore
logous angles are equal,
ach.

triangle D E F be ap-
e triangle A B C so that
E coincide with its
s B, and the side E D
mologous B A; then, be-
homologous angles E and
al to each other [*Dem.*],
F will coincide with its
s B C. Let H and I be the points of B A and B C coin-
h the points D and F; then D F will have the position H I.
e corresponding angles B H I and B A C are equal to each
n.], then H I is parallel to A C [II. 20 i]: therefore B H is
B I is to B C [II. 40] and H I is to A C as B H is to B A
hat is, the homologous sides of the triangles B H I and
proportional. But the triangle B H I is no other than the
E F [*Const.*]; therefore the homologous sides of the tri-
C and D E F are proportional, W. W. T. B. D. Inversely
ngles, the angles of which are not equal, each to each, or the
hich are not proportional, are not similar to each other.



RY I. *The perimeters of two similar triangles are to each*
eir homologous sides (xiv).

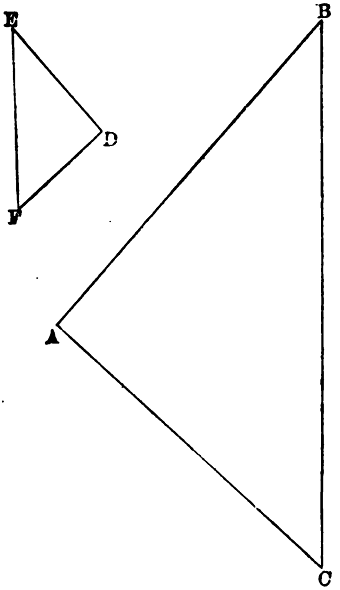
RY II. *If two similar triangles have one side of the one*
e side of the other, they are equal to each other.

THEOREM 23.

(*Eucl.* VI. 4).

Two triangles which have two angles of the one equal angles of the other, each to each, are similar to each ot

Let A B C and D E F be two triangles having the angle to the angle D, and the angle B equal to the angle E.



If the side A B be equ side D E, the two trian equal [16], and consequer lar, to each other.

If A B be not equal tc one of the triangles be until the sides A B anc equal to each other [*Post* the triangle D E F will to the triangle A B C [1(fore they were similar other before the reduct *Def.* 19], W.W.T.B.D. I

If two triangles be no not more than one angle is equal to one angle of th

COROLLARY I. *Two which have the sides oj respectively parallel, or cular, to the sides of the similar to each other.*

COROLLARY II. *Two right-angled triangles which have angle at the hypothenuse, or two isosceles triangles whicl equal angle at the base, are similar to each other.*

COROLLARY III. *All equilateral triangles are similar to one*

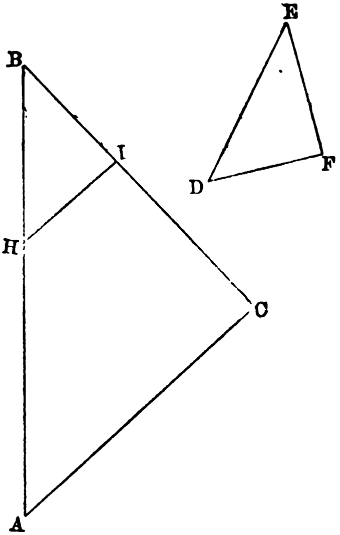
COROLLARY IV. *The triangle formed by the middle lines oj of a triangle is similar to the latter.*

THEOREM 24.

(*Eucl.* VI. 6.)

iangles which have an equal angle formed by pro-
sides, are similar to each other.

C and D E F be two triangles having the angle B equal
e E, and let A B be to D E as B C is to E F.
e equal to D E, the side
ıl to E F [*Gen. Def.* 28]:
the two triangles are
and consequently simi-
ı other.
not equal to DE, let AB
than D E [*Gen. Def.* 28];
ı greater than E F. Let
e D E F be applied to the
B C, so that the vertex
with the vertex B, and
D with the side B A;
ıse the angle E is equal
e B [*Hyp.*], the side E F
.e with the side BC. Let
ə the points of B A and
ding with the points D
ıen the side D F will



ıition H I. Because the lines H I and A C subtend pro-
the angle A B C [*Hyp.*], they are parallel to each other
ence the corresponding angles B H I and B A C are equal
her [II. 20 i]; therefore the triangle A B C is similar to
le H B I [23]. But the triangle H B I is no other than
e D E F [*Const.*]; therefore the triangles A B C and D E F
· to each other, W. W. T. B. D. Inversely
iangles have two sides of the one proportional to two sides of

the other, and if the third sides be not proportional to the first, the angles formed by the proportional sides are unequal.

COROLLARY I. *Two right-angled triangles having proportional perpendicular sides, are similar to each other.*

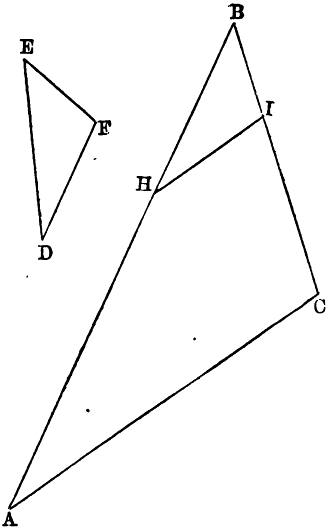
COROLLARY II. *Two isosceles triangles which have an equal angle at the apex, are similar to each other.*

COROLLARY III. *Two isosceles right-angled triangles are similar to each other.*

THEOREM 25.

(*Eucl.* VI. 5.)

Two triangles which have the three sides of the one proportional to the three sides of the other, are similar to each other.



Let A B C and D E F be two triangles such that A B be to D E as B C is to E F and as A C is to D F.

If the side A B be equal to the side D E, the side B C is equal to E F and A C to D F [*Gen. Def.* 28]; therefore the two triangles are equal [18], and consequently similar, to each other.

If A B be not equal to D E, let AB be greater than D E; then BC is greater than E F [*Gen. Def.* 28]. Let B H be equal to E D; let B I be equal to E F, and let H I be joined: then B A is to B H as B C is to B I, and as A C is to D F [*Hyp.*]. Because the lines H I and A C subtend proportionally the angle A B C, they are parallel to each other [II. 39];

e B A is to B H as A C is to H I [II. 40]: but B A is to B H as A C is to D F [*Hyp.*]; then H I is equal to D F herefore the triangles H B I and D E F are equal [18], and ently similar, to each other. Because the lines H I and parallel to each other [*Dem.*], the corresponding angles nd B H I are equal to each other [II. 20 i]; therefore the s A B C and H B I are similar to each other [23]: but ngle H B I is similar to the triangle D E F [*Dem.*]; there- triangles A B C and D E F are similar to each other, . B. D. Inversely

triangles have two sides of the one proportional to two sides of ', and if the angles formed by these sides be not equal to each third sides of the triangles are not proportional to the other

LARY. *Two isosceles triangles which have the base and one side ne, proportional to the base and one side of the other, are o each other.*

um. Equilateral triangles are similar to one another.

THEOREM 26.

(*Eucl.* VI. 7.)

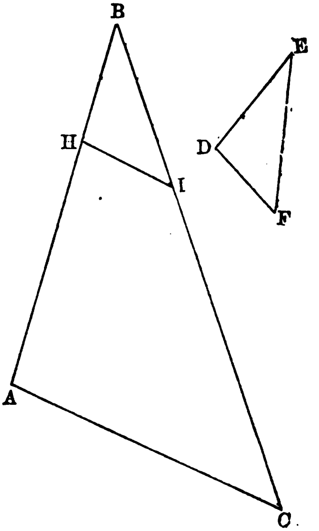
o triangles have one angle of the one equal to one f the other, if the sides forming another angle be onal, and if the third angle be acute in both tri- the two triangles are similar to each other.

B C and D E F be two triangles having the angle B equal to e E; let A B be to D E as A C is to D F, and let the angles be both acute.

side A B be equal to the side D E, the side A C is equal ide D F; therefore the two triangles are equal [20], and ently similar, to each other.

be not equal to D E, let A B be greater than D E; let also qual to D E and let H I be a parallel to A C: then H I is to

A C as B H is to B A [II. 41]. Because D F is to A C as D E is to A B [*Hyp.*], and D E is equal to B H [*Const.*], the line H I is to A C as D F is to A C [i]; therefore H I is equal to D F [xvi]. Because the corresponding angles B I H and B C A are equal to each other [II. 20 i] and the angle B C A is acute, the angle B I H is also acute [i]; therefore the triangles H B I and D E F are equal [20], and consequently similar, to each other. Because the triangles A B C and H B I have the angle B in common, and the angle B C A equal to the angle B I H [*Dem.*], they are similar to each other [23]: but the triangle H B I is similar to the triangle D E F [*Dem.*]; therefore the triangles A B C and D E F are similar to each other, W. W. T. B. D.



COROLLARY I. *If two triangles have one angle of the one equal to one angle of the other, if the sides forming another angle be proportional, and if the third angle be obtuse in both triangles, the two triangles are similar to each other.* (Eucl. VI. 7.)

COROLLARY II. *If two triangles have one angle of the one equal to one angle of the other, and if the sides forming another angle be proportional, the third angle of each triangle is either equal or supplementary to the third angle of the other.*

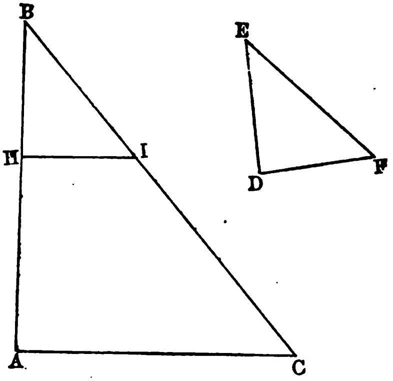
THEOREM 27.

If two right-angled triangles have the hypothenuse and one of the right sides of the one, respectively proportional to the *enuse and to one of the right sides of the other, the angles are similar to each other.*

A B C and D E F be two right-angled triangles, and let the enuse B C be to the hypothenuse E F as the side B A is to E D.

A be equal to E D, then B C is equal to E F; therefore triangles are equal [20], and consequently similar, to each

A be not equal to E D, let B A be greater than E D; then be greater than E F.



H be equal to E D, B I be equal to E F; H I be joined: then to B H as B C is to *p.*]. Because the lines d A C subtend proally the angle A B C, e parallel to each II. 38]; hence the nding angles B A C I are equal to each I. 20 i]: therefore ngle B H I is rightin H. Because the gled triangles B A C and B H I have the angle B in common, e similar to each other [23 ii]; because the right-angled s B H I and D E F have the hypothenuse B I and the side the one respectively equal to the hypothenuse E F and the of the other [*Const.*], they are equal [21], and consequently to each other; therefore the triangles A B C and D E F are o each other, W. W. T. B. D.

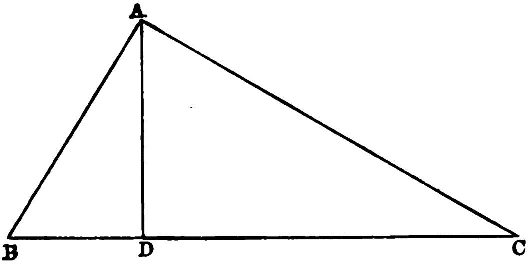
um. Two isosceles right-angled triangles are similar to each iii).

THEOREM 28.

(*Eucl.* VI. 8.)

In every right-angled triangle, the altitude of the vertex of the right angle divides the triangle into two triangles similar to each other.

Let A B C be a right-angled triangle, and let A D be a perpendicular to B C.



The partial triangles D B A and D A C are right-angled in D, and have each an angle at the hypothenuse in common with the triangle A B C; then both are similar to the triangle A B C [23 ii]: therefore they are similar to each other, W. W. T. B. D.

COROLLARY I. *In every right-angled triangle, the altitude of the vertex of the right angle, divides the triangle into two partial triangles similar to the whole.* (Eucl. VI. 8.)

COROLLARY II. *Each one of the perpendicular sides of a right-angled triangle, is a mean proportional between the hypothenuse and its projections on the latter.*

COROLLARY III. *In every right-angled triangle, the altitude of the vertex of the right angle, is a mean proportional between the two portions of the hypothenuse, determined by its foot,* and conversely. (Eucl. VI. 8 i.)

COROLLARY IV. *If the hypothenuse of a right-angled triangle be divided medially by the altitude, the mean is equal to the right side non-adjacent.*

Scholium. The perpendicular to a diameter drawn from a point of the circumference is a mean proportional to the two portions thereof determined by its foot [II. 49 ii].

SECTION IV.

ON QUADRANGLES.

DEFINITIONS.

5. *Opposite* sides, angles, or vertices of a quadrangle are ie which are not adjacent.

quadrangle is designated by the letters at its vertices, or sometimes by etters at two opposite vertices only.

5. The altitude of any vertex of a quadrangle is the per- licular drawn from that vertex to the diagonal joining adjacent vertices.

'. A *lozenge,* or *rhombus,* is a quadrangle the sides of h are equal to one another.

:. A *square* is a lozenge the angles of which are equal ie another.

ice, since the four angles of a square are together equal to four right ; [4], each one is a right angle; that is, each side is perpendicular to ro adjacent ones.

. A *rectangle* is a quadrangle each side of which is endicular to the adjacent one.

ice, a square is a rectangle the sides of which are equal to one er.

. A *parallelogram* is a quadrangle the opposite sides of h are parallel to each other. The *altitude* of a parallelo- ı is the distance of one of its sides from the opposite which is then called the *base.*

ice, a rectangle, or a square, is a parallelogram.

21. Two parallelograms which have two adjacent sides of the one parallel to two adjacent sides of the other, and the diagonals to a common vertex in the same straight line, are said to be *about* that line, and are called *diagonal* parallelograms.

22. A *trapezium* is a quadrangle two sides of which, called the *bases*, are parallel to each other; the other sides are called the *lateral* sides. The *altitude* of a trapezium is the distance of its bases.

23. An *isosceles* trapezium is a trapezium, the lateral sides of which are equal, but not parallel, to each other. A trapezium the lateral sides of which are not equal to each other is termed *scalene*.

24. A *right-angled* trapezium has one of its lateral sides perpendicular to the bases.

THEOREM 29.

In every parallelogram, the opposite sides are equal to each other.

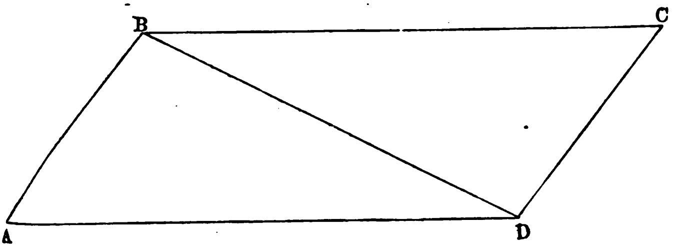
For they are the intercepts of two parallels between two parallels [I. 22].

COROLLARY I. *Each diagonal of a parallelogram divides it into two equal triangles.*

THEOREM 30.

Conversely, *a quadrangle, the opposite sides of which are ual to each other, is a parallelogram.*

Let a quadrangle A C have A B equal to C D and A D equal to C.



Let the diagonal B D be drawn; the quadrangle A C will be then livided into two triangles having the side B D in common. Because he triangles A B D and B D C have the sides A B and A D respectvely equal to the sides C D and C B [29], and the side B D in comnon, they are equal to each other [18]; therefore their homologous ngles are equal to each other. Because the alternate angles A D B nd C B D are equal to each other [*Dem.*], A D is parallel to B C [II. 0i]; because the alternate angles A B D and C D B are equal to each ther, A B is parallel to D C: therefore the quadrangle A C is a arallelogram [*Def.* 20], W. W. T. B. D.

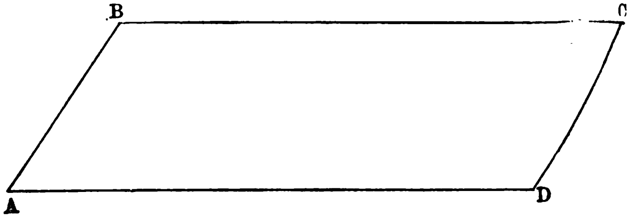
COROLLARY I. *Every lozenge is a parallelogram.*

COROLLARY II. *A quadrangle, two opposite sides of which are equal nd parallel to each other, is a parallelogram.*

THEOREM 31.

The opposite angles of a parallelogram are equal to each ther.

Let A B C D be a parallelogram.



Because the lines A D and B C are parallel to each other, the interior angles A and B are together equal to two right angles [II. 20 ii]; because the lines A B and D C are parallel to each other, the interior angles B and C are equal to two right angles: thence the angles A and B are together equal to the angles B and C [i]; therefore the angles A and C are equal to each other [viii]. The same reasoning would prove that the angles B and D are also equal to each other; therefore the opposite angles of the parallelogram A B C D are equal to each other, W. W. T. B. D.

COROLLARY I. *The opposite angles of a lozenge are equal to each other.*

COROLLARY II. *A parallelogram of which two adjacent angles are equal to each other, is a rectangle.*

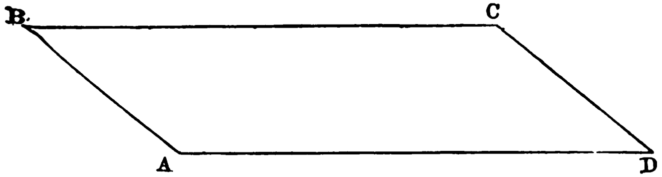
COROLLARY III. *If two parallelograms have an angle of the one equal to one angle of the other, the remaining angles of the first are equal, each to each, to the remaining angles of the second.*

Scholium. The lozenge of which two adjacent angles are equal to each other, is a square.

THEOREM 32.

Conversely, *the quadrangle of which the opposite angles are equal to each other, is a parallelogram.*

Let A B C D be a quadrangle of which the angle A is equal to the angle C, and the angle B equal to the angle D.



Because the angles A and B are respectively equal to the angles C and D, the angles A and B are together equal to the angles C and D [iv]. Because the four angles A, B, C and D are together equal to four right angles [4], the angles A and B are together equal to two right angles [vii]: therefore the lines B C and A D are parallel to each other [II. 21 ii]. The same reasoning would prove that the lines A B and D C are parallel to each other; therefore the quadrangle A B C D is a parallelogram, W. W. T. B. D.

COROLLARY. *The quadrangle of which two opposite sides are parallel and two opposite angles equal to each other, is a parallelogram.*

THEOREM 33.

The diagonals of a parallelogram bisect each other.

For they join the unlike extremities of two equal parallels [I. 25 i].

COROLLARY I. Conversely, *the quadrangle of which the diagonals bisect each other, is a parallelogram.*

COROLLARY II. *The diagonals of a rectangle are equal to each other.*

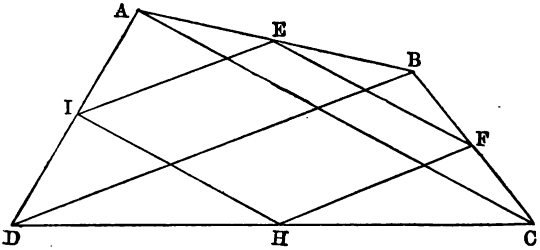
COROLLARY III. *The diagonals of a lozenge are the middle perpendicular on each other.*

COROLLARY IV. *The intersection of the diagonals of a parallelogram, bisects every straight line drawn through it and terminating at the perimeter.*

THEOREM 34.

The middle lines of the adjacent sides of every quadrangle form a parallelogram.

Let E F, F H, H I and I E be the middle lines of the adjacent sides of the quadrangle A B C D.



Let A C and B D be the diagonals of the quadrangle. Because E F and I H are parallel to A C [11 i], they are parallel to each other [I. 17 iii]; because E I and F H are parallel to B D, they are parallel to each other: therefore the quadrangle E F H I is a parallelogram, W. W. T. B. D.

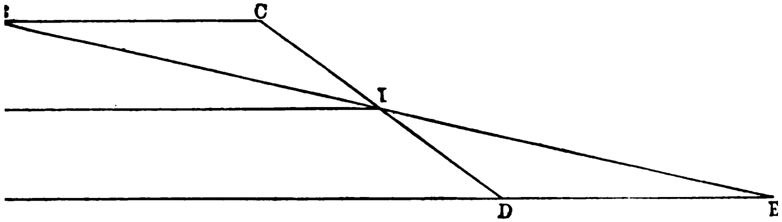
COROLLARY I. *The middle lines of the opposite sides of a quadrangle bisect each other.*

COROLLARY II. *The middle lines of the adjacent sides of a rectangle form a lozenge; those of the adjacent sides of a lozenge form a rectangle; and those of the adjacent sides of a square form a square.*

THEOREM 35.

The middle line of the lateral sides of a trapezium, is equal to half the sum of the bases.

Let A B and C D be the lateral sides of the trapezium A C, and let H I be their middle line.



Let B I be joined, and let A D and B I be produced until they meet in E. Because C I is equal to I D [*Hyp.*], and the vertical angles B I C and D I E are equal to each other [II. 3], and also the alternate angles B C I and I D E are equal to each other [II. 20], the triangles B I C and D I E are equal to each other [15]: therefore D E is equal to B C; that is, A E is the sum of A D and B C; also I E is equal to B I. Because H I is the middle line of B A and B E, it is equal to half of A E [11]: therefore H I is equal to half the sum of the bases A D and B C, which W. T. B. D.

COROLLARY I. *The middle line of the diagonals of a trapezium is equal to half the difference of the bases.*

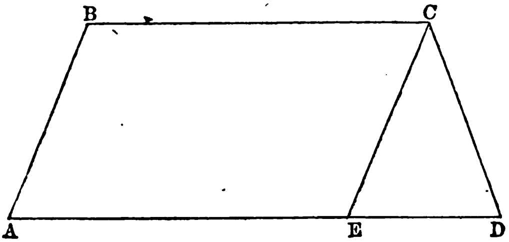
COROLLARY II. *A parallel to the bases of a trapezium divides the lateral sides proportionally* [II. 38 iii].

THEOREM 36.

The angles at each base of an isosceles trapezium are equal to each other.

Let A B C D be an isosceles trapezium, and let A D and B C be its bases.

Let C E be a parallel to B A; then A B C E is a parallelogram. Because C E is equal to A B [29], and A B is equal to C D [*Hyp.*], the line C E is equal to C D [i]; therefore the angle C E D is equal to the angle C D E [8]: now, the corresponding angles B A D and



C E D are equal to each other [II. 20 i]; therefore the angle B A D is equal to the angle C D A [i].

Because the angles A B C and D C B of the trapezium are respectively supplementary to the angles A and D [II. 20 ii], and because the angles A and D are equal to each other [*Dem.*], the angle A B C is equal to the angle D C B [II. 14]: therefore the angles at each base of the isosceles trapezium A C, are equal to each other, W. W. T. B. D.

Corollary I. Conversely, *if the angles at one base of a trapezium be equal to each other, the trapezium is isosceles.*

Corollary II. *The diagonals of an isosceles trapezium are equal to each other.*

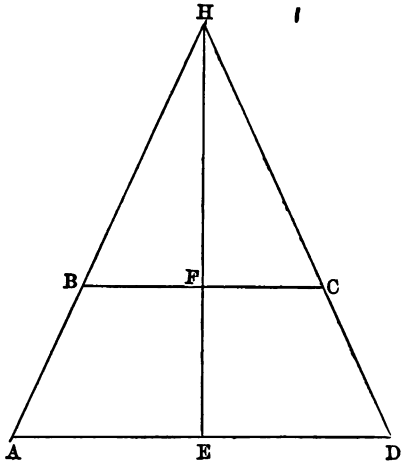
Theorem 37.

The middle line of the bases of an isosceles trapezium is their middle perpendicular.

Let E F be the middle line of the bases of the isosceles trapezium A B C D.

Let the sides A B and D C be indefinitely produced; then they will meet in a point H [I. 18 i]. Because the triangle A H D has the

A equal to the angle D, the side A H is equal to D H [7];
B is equal to D C [*Hyp.*], therefore H B is equal to H C [iv].
E F be produced.
e A D and B C
d symmetrically
gle A H D, the
, indefinitely pro-
will pass through
int H and bisect
le A H D [II. 8 i];
re E F is perpen-
to A D and B C
which W. T. B. D.



OLLARY I. *The*
ngle which has the
line of the oppo-
s for their middle
icular, is an iso-
apezium.

LLARY II. *The lateral sides of an isosceles trapezium are sym-*
o the middle line of the bases.

LLARY III. *The middle line of the bases of an isosceles trape-*
ivides it into two equal right-angled trapeziums.

LLARY IV. *The middle points of the bases of an isosceles*
m, the meeting point of the lateral sides and the intersection
iagonals, are points of the same straight line.

SECTION V.

ON THE COMPARISON OF POLYGONS.

DEFINITIONS.

25. The *homologous* sides, angles, vertices, or diag of two equal, or similar, polygons are those which coi when the two polygons, being reduced to the same ma tude, are made to coincide. The *homologous* triangles o equal, or similar, polygons, are those which are forme homologous sides, or diagonals.

26. Two polygons are *equilateral to each other*, or si *equilateral*, when the sides of the one are equal, ea each, to the sides of the other; the corresponding sides are called the *homologous* sides of the two polygo

Hence, two equal polygons are equilateral to each other, for wh two polygons coincide, their sides coincide, and consequently are equ to each.

It is obvious that two equilateral triangles, or two squares, hav equal side, are equilateral to each other, and that their homologou may be chosen indiscriminately.

27. Two polygons are *equiangular to each other*, or si *equiangular*, when the angles of the one are equal, ea each, to the angles of the other; the corresponding angles are called the *homologous* angles of the two poly

Hence, two parallelograms having an equal angle are equiangular other [31 iii].

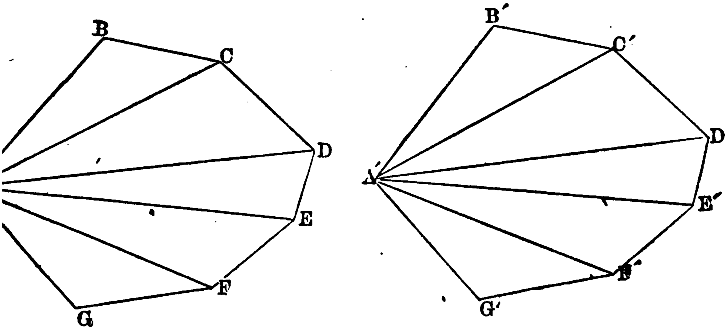
It is evident that two equal polygons are equiangular to each other, and lso that two equiangular polygons, the homologous angles of which are rmed by sides equal each to each, are equal to each other, for they may be ade to coincide.

All rectangles and squares are obviously equiangular to each other, and he homologous angles may be chosen indiscriminately.

THEOREM 38.

Two equal polygons may be divided into the same number f triangles equal each to each.

Let A B C D E F G and A′ B′ C′ D′ E′ F′ G′ be two equal polygons f which the vertices A′, B′, C′, etc., are respectively homologous to he vertices A, B, C, etc.



Let the diagonals A C, A D, etc., and A′ C′, A′ D′, etc., be drawn; en the two polygons will both be divided into the same number of iangles [2i]. Now, let the two polygons be applied to each other that the homologous vertices coincide, then the triangles A B C, C D, etc., will coincide with the triangles A′ B′ C′, A′ C′ D′, etc.; erefore the triangles A B C, A C D, etc., are equal, each to each, to e triangles A′ B′ C′, A′ C′ D′, etc.; which W. T. B. D.

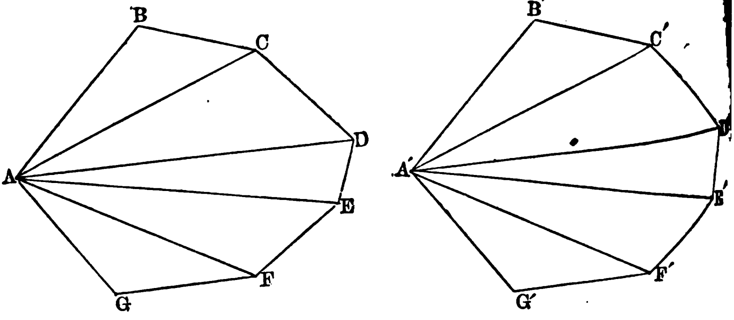
COROLLARY I. *In equal polygons, the homologous diagonals are equal to each other.*

COROLLARY II. *The homologous triangles of equal polygons are equal to each other.*

THEOREM 39.

Conversely, *if two polygons be composed of the same number of triangles equal each to each and similarly placed, they are equal to each other.*

Let the polygons A B C D E F G and A′ B′ C′ D′ E′ F′ G′ be composed of the same number of triangles A B C, A B D, etc., and A′ B′ C′, A′ C′ D′, etc.; let the triangles composing one of the polygons be equal, each to each, to the triangles composing the other polygon.



Let the two polygons be applied to each other, so that the equal triangles A B C and A′ B′ C′ coincide; then the diagonal A C will coincide with the diagonal A′ C′, and the angle A C D will coincide with its equal A′ C′ D′: therefore the triangle A C D will coincide with the triangle A′ C′ D′. The same reasoning will prove that all ngles composing the polygon A B C D E F G, will coincide ose composing the polygon A′ B′ C′ D′ E′ F′ G′; therefore they l to each other, W. W. T. B. D.

LARY I. *Two squares having an equal side are equal to each*

LARY II. *Two rectangles which have two adjacent sides of the l to two adjacent sides of the other, each to each, are equal to er.*

LARY III. *Two parallelograms having an equal angle formed equal each to each, are equal to each other.*

LARY IV. *Two trapeziums which have the sides of the one the sides of the other, each to each, are equal to each other.*

THEOREM 40.

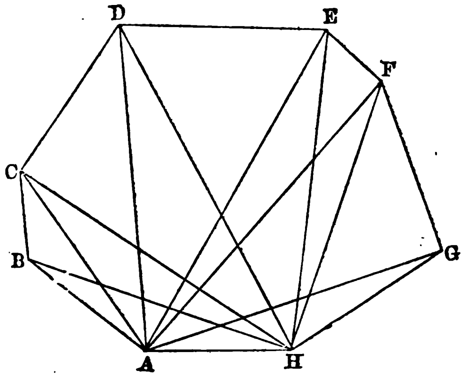
vo polygons have the angles formed by one side of the h the adjacent sides and diagonals equal, each to each, angles formed by an equal side of the other with the t sides and diagonals, they are equal to each other.

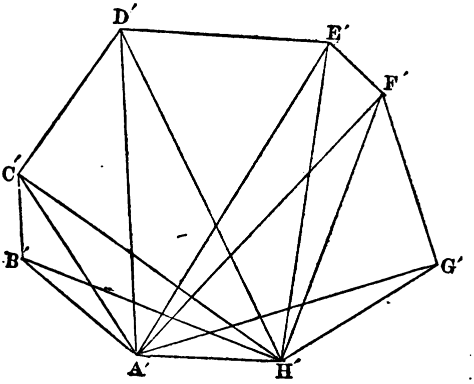
e side A H of the polygon A B C D E F G H be equal to the H′ of the polygon A′ B′ C′ D′ E′ F′ G′ H′; let also the angles by the side A H with the adjacent sides and diagonals be ach to each, to the angles formed by the side A′ H′ with the t sides and diagonals.

se the triangles A B H and A′ B′ H′ have the sides A H and ual to each other, the angle B A H equal to the angle B′ A′ H′, angle B H A equal to the angle B′ H′ A′ [*Hyp.*], they are equal other [16]; therefore the side A B is equal to the side A′ B′.

se the triangles A C H and A′ C′ H′ have the side A H equal side A′ H′, the angle C A H equal to the angle C′ A′ H′, and le C H A equal to the angle C′ H′ A′ [*Hyp.*], they are equal to her [16]; then the diagonal A C is equal to the diagonal A′ C′.

Now, the triangles A B C and A′ B′ C′ have the angle B A C equal to the angle B′ A′ C′ [*Hyp.*], the side A B equal to the side A′ B′, and the side A C equal to the side A′ C′ [*Dem.*]; therefore they are equal to each other [17].





Because the triangles A D H and A′ D′ H′ have the side A H equal to the side A′ H′, the angle D A H equal to the angle D′ A′ H′, and the angle D H A equal to the angle D′ H′ A′ [*Hyp.*], they are equal to each other [16]; then the diagonal A D is equal to the diagonal A′ D′. Now, the triangles A C D and A′ C′ D′ have the angle C A D equal to the angle C′ A′ D′ [*Hyp.*], the side A C equal to the side A′ C′, and the side A D equal to the side A′ D′ [*Dem.*]; therefore they are equal to each other [17].

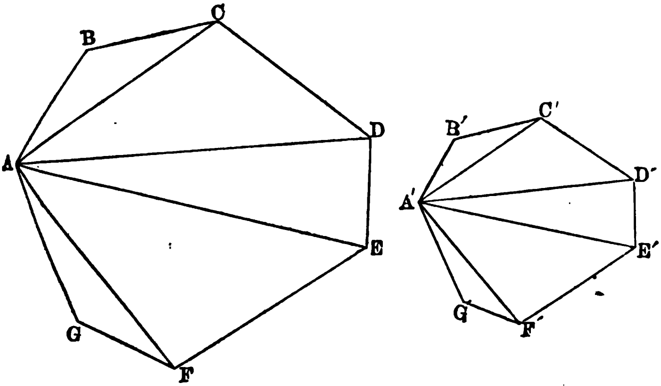
The same reasoning would prove that the remaining triangles of the polygon A B C D E F G H are equal, each to each, to the remaining triangles of the polygon A′ B′ C′ D′ E′ F′ G′ H′; that is, the two polygons are composed of the same number of triangles equal each to each and similarly placed: therefore they are equal to each other [39], W. W. T. B. D.

THEOREM 41.

(*Eucl.* VI. 20.)

Two similar polygons may be divided into the same number of triangles similar each to each.

Let A B C D E F G and A′ B′ C′ D′ E′ F′ G′ be two similar polygons.



If the two polygons be of the same magnitude,—they are equal to each other [*Gen. Def.* 18]: therefore they may be divided into the same number of triangles equal, and consequently similar, each to each [38].

If the two polygons be not of the same magnitude, let the vertex A and its homologous A′ be joined to the non-adjacent vertices; then the two polygons will be divided into the same number of triangles [22]. Let the two polygons be reduced to the same magnitude; then they are equal to each other [*Gen. Def.* 18]. Now, let the two polygons be applied to each other so that the homologous vertices coincide; they will then coincide with each other as also the triangles into which they are divided: therefore the triangles of the one are equal, each to each, to those of the other. But, in the reduction of the polygons, the partial triangles have also been reduced [*Post.* I]; therefore the triangles into which one of the polygons is divided were, before the reduction, similar each to each, to the triangles into which the other polygon is divided, W. W. T. B. D.

Corollary I. *Two similar polygons are equiangular to each other.*

Corollary II. *The homologous sides and diagonals of two similar polygons are proportional.*

Corollary III. *The perimeters of two similar polygons are to each other, as their homologous sides or diagonals* (xiv).

Corollary IV. *If two similar polygons have one side of the one equal to one side of the other, they are equal to each other.*

Corollary V. *The homologous triangles of similar polygons are similar to each other.*

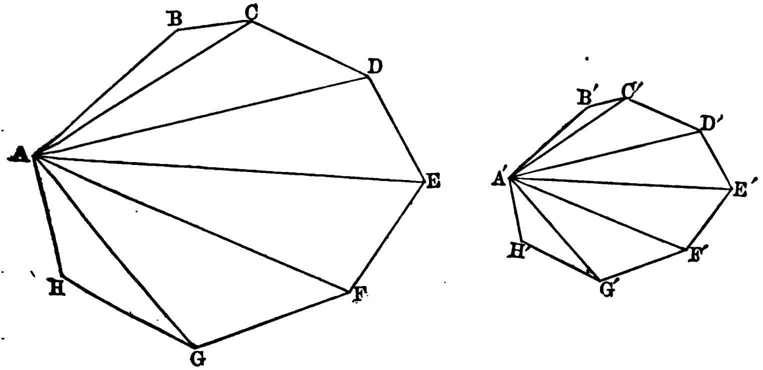
Theorem 42.

Conversely, *if two polygons be composed of the same number of triangles similar each to each and similarly placed, they are similar to each other.*

Let the polygons A B C D E F G H and A′ B′ C′ D′ E′ F′ G′ H′ be composed of the same number of triangles A B C, A C D, etc., and A′ B′ C′, A′ C′ D′, etc., similar each to each.

If A B be equal to A′ B′, then the triangles A B C and A′ B′ C′ are equal to each other [22 ii], and A C is equal to A′ C′; therefore the triangles A C D and A′ C′ D′ are equal to each other. The same reasoning would prove that the remaining triangles of one polygon are equal, each to each, to the remaining triangles of the other; therefore the two polygons are equal [39], and consequently similar, to each other.

If A B be not equal to A′ B′, let the greater polygon be reduced until the similar triangles A B C and A′ B′ C′ have the same magnitude [*Post.* I]; these two triangles will then be equal to each other [*Gen. Def.* 18]. Now, let the reduced polygon be applied to the other, so that the equal triangles A B C and A′ B′ C′ coincide with each other; the angles C A D, D A E, etc., will coincide with their equals C′ A′ D′, D′ A′ E′, etc. [22]; also the diagonal A C will coincide with the diagonal A′ C′. Because the similar triangles A C D and A′ C′ D′ have now the side A C equal to the side A′ C′, they are equal to each other [22 ii]: therefore they coincide with each other, and the diagonal A D is now equal to the diagonal A′ D′. Likewise the remaining triangles of one polygon coincide respectively with the remaining triangles of the other; hence the two polygons coincide with each other; therefore they are now equal to each other: but they have been reduced to the same magnitude [*Const.*]; therefore they were similar before the reduction, W. W. T. B. D.



Corollary I. *Two squares are similar to each other.*

Corollary II. *Two rectangles having two adjacent sides proportional, are similar to each other.*

angles into which they are divided: therefore the triangles of th one are equal, each to each, to those of the other. But, in th reduction of the polygons, the partial triangles have also been re duced [*Post.* I]; therefore the triangles into which one of th polygons is divided were, before the reduction, similar each t each, to the triangles into which the other polygon is divided W. W. T. B. D.

Corollary I. *Two similar polygons are equiangular to each other.*

Corollary II. *The homologous sides and diagonals of two simil polygons are proportional.*

Corollary III. *The perimeters of two similar polygons are to ea other, as their homologous sides or diagonals* (xiv).

Corollary IV. *If two similar polygons have one side of the on equal to one side of the other, they are equal to each other.*

Corollary V. *The homologous triangles of similar polygons ar similar to each other.*

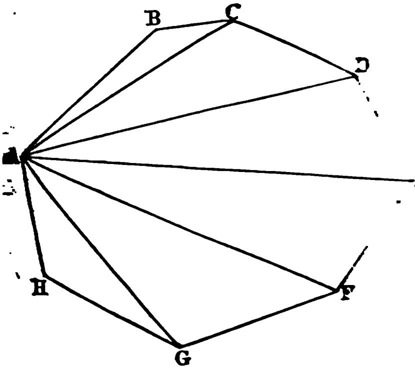
Theorem 42.

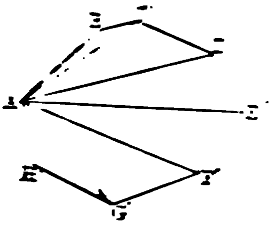
Conversely, *if two polygons be composed of the same number of triangles similar each to each and similarly placed, they are similar to each other.*

Let the polygons A B C D E F G H and A′ B′ C′ D′ E′ F′ G′ H′ be composed of the same number of triangles A B C, A C D, etc., and A′ B′ C′, A′ C′ D′, etc., similar each to each.

If A B be equal to A′ B′, then the triangles A B C and A′ B′ C′ are equal to each other [22 ii], and A C is equal to A′ C′; therefore the triangles A C D and A′ C′ D′ are equal to each other. The same reasoning would prove that the remaining triangles of one polygon are equal, each to each, to the remaining triangles of the other; therefore the two polygons are equal [39], and consequently similar, to each other.

If A B be not equal to A′ B′, let the greater polygon be reduced until the similar triangles A B C and A′ B′ C′ have the same magnitude



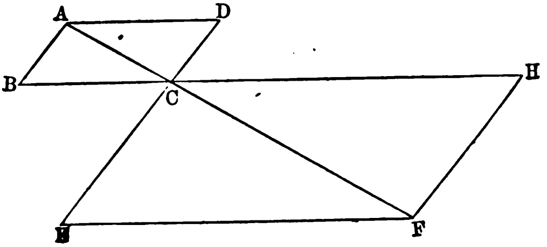


tude [*Post.* I]; these two triangles will then be equal to each other [*Gen. Def.* 18]. Now, let the reduced polygon be applied to the other, so that the equal triangles A B C and A′ B′ C′ coincide with each other; the angles C A D, D A E, etc., will coincide with their equals C′ A′ D′, D′ A′ E′, etc. [22]; also the diagonal A C will coincide with the diagonal A′ C′. Because the similar triangles A C D and A′ C′ D′ have now the side A C equal to the side A′ C′, they are equal to each other [22 ii]: therefore they coincide with each other, and the diagonal A D is now equal to the diagonal A′ D′. Likewise the remaining triangles of one polygon coincide respectively with the remaining triangles of the other; hence the two polygons coincide with each other; therefore they are now equal to each other: but they have been reduced to the same magnitude [*Const.*]; therefore they were similar before the reduction, W. W. T. B. D.

COROLLARY I. *Two squares are similar to each other.*

COROLLARY II. *Two rectangles having two adjacent sides proportional, are similar to each other.*

vertex C in common; let the sides A D and A B be parallel to the sides C H and C E respectively, and let the diagonals A C and C F be the continuation of each other.



Because C B and C H are each parallel to D A [*Hyp.*], they are parallel to each other [I. 17], and C H and C B are the continuation of each other [I. 17 iv]: likewise C D and C E are the continuation of each other; therefore the angles B C D and E C H are equal to each other [II. 2].

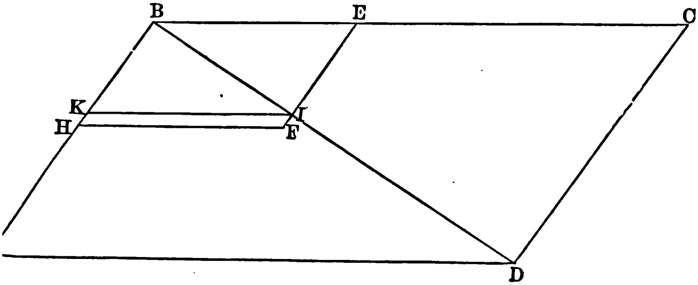
Because the triangles A D C and C H F have the sides of the one parallel to the sides of the other, they are similar to each other [23 i]; therefore A D is to C H as D C is to F H, [22]: but C B and C E are respectively equal to A D and F H [29]; then C B is to C H as C D is to C E; therefore the parallelograms B D and E H are similar to each other [42 iii], W. W. T. B. D.

COROLLARY. *If a parallelogram be divided into four partial parallelograms by parallels to two adjacent sides through a point of one diagonal, the two partial parallelograms which are about the diagonal, are similar to the whole.*

THEOREM 45.

If two similar parallelograms similarly placed have an angle in common, they are about the same straight line.

Let A B C D and B E F H be two similar parallelograms similarly
ced and having the angle A B C in common.



Let B D be a diagonal of the parallelogram A C and let I be the
int where B D cuts E F; let I K be parallel to E B: then the
rallelograms K E and A C are about the line B D [*Def.* 21]. Be-
use the parallelograms K E and A C are similar to each other [44],
C is to B E as B A is to B K [41 ii]. But the parallelograms H E
d A C are also similar to each other [*Hyp.*], then B C is to B E as
A is to B H: therefore B K is equal to B H [xvii]; that is, H and K
e one and the same point. Because I K and F H are parallel to E B,
ey are parallel to each other [I. 17 iii], and coincide with each
her [I. 17 iv]: that is, the parallelograms K E and H E are one and
e same figure; therefore the parallelograms H E and A C are
out the line B D [*Def.* 21], which W. T. B. D.

COROLLARY. *If two similar parallelograms similarly placed have
tical angles, they are about the same straight line.*

CHAPTER II.

ON SYMMETRIC PLANE FIGURES.

SECTION I.

ON SYMMETRIC POLYGONS.

DEFINITIONS.

28. Two plane figures are *symmetric,* or *placed symmetrically, to an axis* when the extremity of every perpendicular drawn from the perimeter of one of them, to a straight line called their *line,* or *axis, of symmetry,* and produced beyond it to a distance equal to itself, is a point of the perimeter of the other; each of the two figures is the symmetric of the other; two points of their perimeters are termed *symmetric* when the line of symmetry is the middle perpendicular on the straight line joining them.

29. A plane figure is *symmetric to an axis* when all points of its perimeter on the same perpendicular to an interior straight line, called *axis of symmetry,* or simply *axis,* are equally distant from that line: two points of the perimeter which are on the same perpendicular to the axis are termed *symmetric.*

It follows from this definition that if one of the sides of a polygon symmetric to an axis be met by the axis, the latter is the middle perpendicular on that side.

30. Two sides, or diagonals of a polygon, or two polygons symmetric to an axis, are *symmetric* sides, or diagonals when the extremities of one of them are the symmetrics of those of the other; two angles are termed *symmetric,* when the sides of one of them are the symmetrics of those of the other.

From these definitions it follows that in a plane figure, or two plane figures, symmetric to an axis, their symmetric points, lines and angles, are symmetric to the axis [I. 29].

31. Two plane figures are *symmetric,* or *placed symmetrically, to a point* when the extremities of every straight line drawn from the perimeter of one of them to a point, called their *point of symmetry,* and produced beyond it to a distance equal to itself, is a point of the perimeter of the other; each one of the two figures is the *opposite* of the other: two points of their perimeters are *opposite* points of the figures when the point of symmetry bisects the straight line joining them; each of the two points is the *opposite* of the other.

32. A plane figure is *symmetric to a point* when all points of the perimeter on a straight line with an interior point, called the *centre of symmetry,* or simply *centre,* are equally distant from that point; two points of the perimeter which are on the same straight line with the centre, are *opposite* points of the figure.

33. Two *opposite* sides, or diagonals, of polygons, symmetric to a point, are those in which the extremities of the one are opposite to those of the other; two *opposite* angles have opposite vertices and sides.

34. A *diameter* of a polygon symmetric to a point is the line joining two opposite vertices.

Hence, each diameter passes through the centre.

THEOREM 46.

The perimeters of two polygons symmetric to an axis, are symmetric to that axis.

Let there be two polygons P and Q symmetric to an axis.

Let the plane containing them be folded upon the axis; then every point of the perimeter of each will coincide with its symmetric in the other [I. 9 i]: that is, the two perimeters will coincide throughout; therefore these perimeters are symmetric to the axis, W. W. T. B. D.

COROLLARY I. *The perimeter of a polygon symmetric to an axis, is divided by that axis into two parts symmetric to it.*

COROLLARY II. *Two polygons symmetric to an axis are equal to each other.*

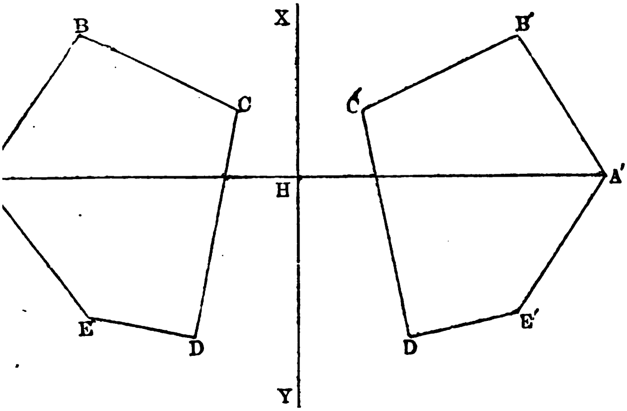
COROLLARY III. *A polygon symmetric to an axis is divided by the latter into two equal parts.*

THEOREM 47.

Conversely, *if the perimeters of two polygons be symmetric to a straight line, the polygons which they limit are symmetric to the same line.*

Let the perimeters of the two polygons A B C D E and A′ B′ C′ D′ E′ be symmetric to the straight line X Y.

Let A and A′ be two points of their perimeters symmetric to the line X Y, and let A A′ be joined; then X Y is the middle perpendicular on A A′ [I. 28]: therefore the extremity of the perpendicular drawn from the point A on X Y, and produced beyond its foot H to a ce equal to its length, will coincide with the point A′. The construction and reasoning would prove that the extremity



e perpendiculars drawn from every point of the perimeter of f the polygons A B C D E and A′ B′ C′ D′ E′ on X Y, and pro- l beyond its foot to a distance equal to its length, will coincide a point of the perimeter of the other; therefore these two ons are symmetric to X Y, which W. T. B. D.

ROLLARY I. *A polygon is symmetric to the straight line which es its perimeter into two parts symmetric thereto.*

ROLLARY II. *An isosceles or equilateral triangle is symmetric to itude; a lozenge, to its diagonals; a rectangle, to the middle lines opposite sides; an isosceles trapezium, to the middle line of its*

ROLLARY III. *The extremities of the perpendiculars drawn from the es of a polygon to an exterior straight line, and produced beyond it istance equal to itself, are the vertices of a polygon symmetric of st.*

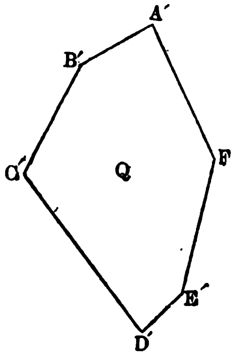
ROLLARY IV. *The extremities of any number of straight lines having me middle perpendicular, are the vertices of a polygon symmetric latter.*

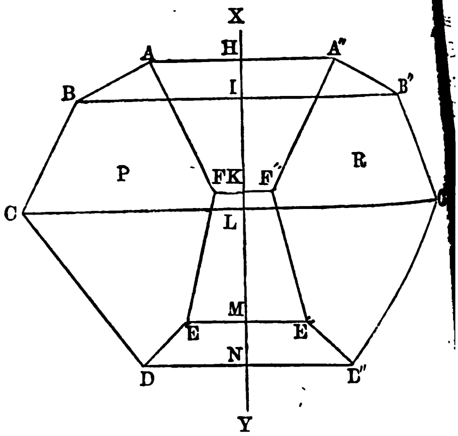
Scholium. An equilateral triangle has three, a lozenge or rectangle has two, and a square four, axes of symmetry.

THEOREM 48.

Two equal polygons may be placed symmetrically to an axis.

Let the polygons P and Q be equal to each other, and let the vertices A′, B′, C′, etc., of the polygon Q′ be homologous to the vertices A, B, C, etc., of the polygon P respectively.





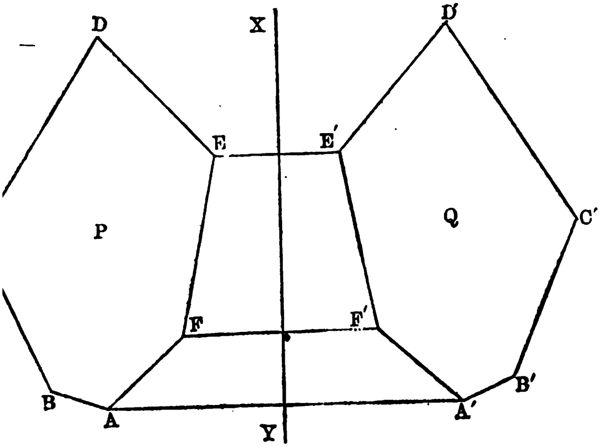
Let there be an indefinite straight line X Y, and let there be from the vertices A, B, C, etc., perpendiculars A H, B I, C L, etc., to X Y; let them be produced beyond that line to distances H A″, I B″, L C″, etc., equal to their respective lengths: then the points A″, B″, C″, etc., are the symmetrics of the points A, B, C, etc. [I. 29]. Let A″B″, B″C″, C″D″, etc., be joined, and let R be the polygon thus formed; then the polygons P and R are symmetric to X Y [47].

the polygon P is equal to the polygon Q [*Hyp.*] and to the R [46 ii], the polygons Q and R are equal to each other; ey may be made to coincide: therefore, the polygons P and Q laced symmetrically, W. W. T. B. D.

THEOREM 49.

qual polygons are symmetric to the straight line which iddle perpendicular on three lines joining their homo- ertices.

e polygons P and Q be equal to each other, and let the ver- B′, C′, etc., of the polygon Q be homologous to the vertices etc., of the polygon P respectively; let A A′, F F,′ and E E′ e straight lines joining their homologous vertices, and let the middle perpendicular thereon.



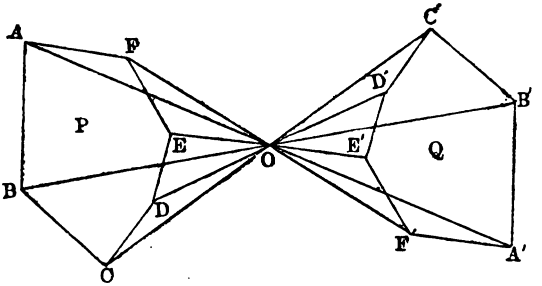
he plane containing the two polygons be folded upon X Y; e points A, F, and E will coincide with their symmetrics

A′, F′ and E′ respectively [I. 29]. Because the homologous angles A and A′ are equal to each other [*Hyp.*], the side A B will coincide with the side A′ B′; and because the homologous sides A B and A′ B′ are equal to each other [*Hyp.*], the vertex B will coincide with the vertex B′. The same reasoning applied to the remaining angles and sides, would prove that the perimeters of the polygons P and Q will coincide with each other; that is, they are symmetric to X Y: therefore the polygons P and Q are symmetric to X Y [47], which W. T. B. D.

THEOREM 50.

Opposites sides of two polygons symmetric to a point are parallel and equal to each other.

Let there be two polygons P and Q symmetric to a point O, and let the vertices A, B, C, etc., of the polygon P, be opposite to the vertices A′ B′, C′, etc., of the polygon Q respectively.



Let A A′, B B′, C C′, etc., be joined: then each of them is bisected in the point O [*Def.* 31]. Because the sides of the angle A O B and its vertical are cut proportionally by A B and A′ B′, the line A B is parallel to A′ B′ [II. 39 i]; and because O A is equal to O′ A′, the line A B is equal to A′ B′ [II. 41 ii]. The same reasoning would prove that

B C, C D, etc., are parallel and equal to B′ C′, C′ D′, etc., respectively; therefore the opposite sides of the polygons P and Q are parallel and equal to each other, W. W. T. B. D.

Corollary I. *The opposite sides of a polygon symmetric to a point, are parallel and equal to each other.*

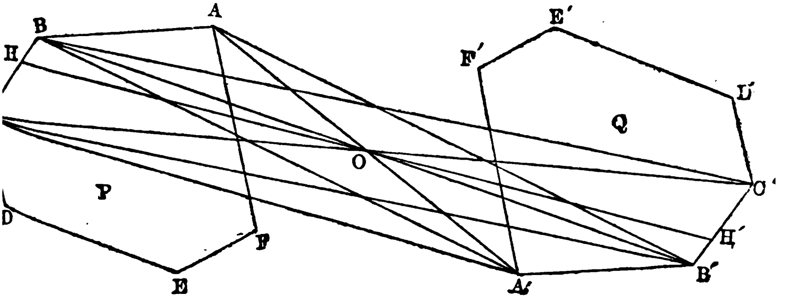
Corollary II. *Opposite diagonals of two polygons, or of a polygon symmetric to a point, are parallel and equal to each other.*

Corollary III. *Opposite angles of two polygons, or of a polygon symmetric to a point, are equal to each other.*

Theorem 51.

Conversely, *if two polygons have the sides of the one parallel and equal to the sides of the other, each to each, the two polygons are symmetric to the intersection of the lines joining the unlike extremities of two corresponding parallel sides.*

Let the sides A B, B C, C D, etc., of the polygon P be equal and parallel respectively to the sides A′ B′, B′ C′, C′ D′, etc., of the polygon Q.



Let A A′, B B′, be joined, and let O be their intersection; let also A B′ and B A′ be joined. Because A B and A′ B′ are equal and parallel to each other [*Hyp.*], the quadrangle A B B′ A′ is a parallelogram [30 ii]; hence A O is equal to O A′ and B O is equal to O B′ [33].

Let C C′, B C′ and C B′ be joined. Because B C and B′ C′ are equal and parallel to each other [*Hyp.*], the quadrangle B C C′ B′ is a parallelogram [30 ii]; hence B B′ and C C′ bisect each other [33]: but O is the middle point of B B′; therefore C C′ cuts B B′ in O, and C O is equal to O C′. The same construction and reasoning would prove that straight lines joining D to D′, E to E′, etc., would bisect each other in O.

Let any point H of the perimeter of the polygon P be joined to the point O, and produced to a point H′ of the perimeter of the polygon Q. Because the quadrangle B C C′ B′ is a parallelogram [*Dem.*], and H H′ passes through the intersection of its diagonals, H O is equal to O H′ [33 iv]; therefore the extremity of H O produced to a distance equal to its length, is a point of the perimeter of the polygon Q. The same construction and reasoning would prove, that the extremity of the straight line joining any other point of the perimeter of one of the two polygons P and Q to the point O, and produced to a distance equal to its length, is a point of the perimeter of the other: therefore the two polygons are symmetric to the point O, which W. T. B. D.

COROLLARY I. *If each side of a polygon be equal and parallel to another side thereof, the polygon is symmetric to the intersection of two diagonals.*

COROLLARY II. A *parallelogram is symmetric to the intersection of its diagonals.*

COROLLARY III. *The extremities of the straight lines drawn from the vertices of a polygon to an exterior point, and produced beyond it to a distance equal to their respective lengths, are the vertices of a polygon opposite to the first.*

COROLLARY IV. *The extremities of any number of straight lines bisecting each other, are the vertices of a polygon symmetric to their intersection.*

Scholium. A rectangle, square, or lozenge, is symmetric to the intersection of its diagonals.

THEOREM 52.

Two polygons symmetric to a point are equal to each other.

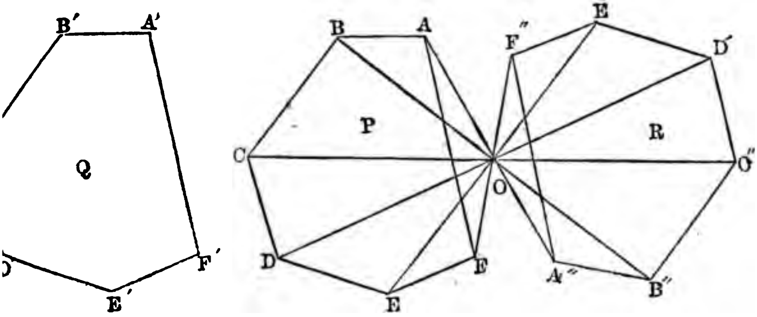
For their opposite angles are formed by parallel sides, equal each to each [59] and having opposite directions.

COROLLARY. *Every diameter of a polygon symmetric to a point, divides it into two equal parts.*

THEOREM 53.

Two equal polygons may be placed symmetrically to a point.

Let there be two polygons P and Q equal to each other, and let the vertices A′, B′, C′, etc., of the polygon Q be homologous to the vertices A, B, C, etc., of the polygon P respectively.



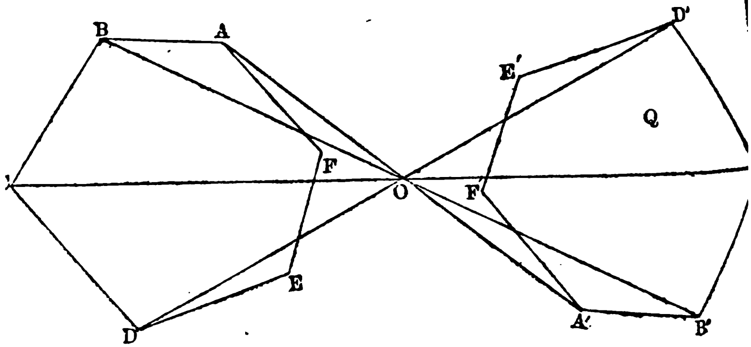
Let the vertices A, B, C, etc., of the polygon P be joined to any point O, and let A O, B O, C O, etc., be produced beyond O to A″, B″, C″, etc., so that O A″, O B″, O C″, be equal to A O, B O, C O, etc., etc., respectively; also let A″ B″, B″ C″, C″ D″, etc., be joined, and let R be the polygon thus formed: then the polygons P and R are symmetric to the point O [51 iii]. Because the polygon P is equal to the polygon Q [*Hyp.*] and to the polygon R [52], the polygons Q and R are equal to each other [i]; hence they may be made to coincide; therefore the polygons P and Q may be placed symmetrically to a point, W. W. T. B. D.

COROLLARY. *Two polygons symmetric to an axis may be placed symmetrically to a point.*

THEOREM 54.

If three straight lines joining the homologous vertices of two equal polygons bisect each other, the two polygons are symmetric to their intersection.

Let the polygons P and Q be equal to each other; let the vertices A′, B′, C′, etc., of the polygon Q be joined to the homologous vertices A, B, C, etc., of the polygon P, and let the lines A A′, B B′, and C C′ bisect each other in O.



Because A B and A′ B′ subtend proportionally the sides of the angle A O B and its vertical, A B is parallel to A′ B′ [II. 39 i]; because B C and B′ C′ also subtend proportionally the sides of the angle B O C and its vertical, B C is parallel to B′ C′; hence the rnate angles B C O and B′ C′ O are equal to each other [II. : but the angle B C D is equal to its homologous B′ C′ D′ p.]; then the alternate angles O C D and O C′ D′ are equal to each r [iv]: therefore C D is parallel to C′ D′ [II. 21].

ecause the angle C D E is equal to its homologous C′ D′ E′ [*Hyp.*], the alternate angles D′ D C and D D′ C′ are equal to each other 20], the alternate angles D′ D E and D D′ E′ are equal to each r [iv]; therefore D E is parallel to D′ E′ [II. 21]. The same struction and reasoning would prove that the remaining sides of polygon P are parallel to the remaining sides of the polygon Q ectively: thus the homologous sides of the polygons P and Q are al [*Hyp.*] and parallel [*Dem.*] to each other; therefore, the two gons are symmetric to the point O [51], which W. T. B. D.

SECTION II.

ON REGULAR POLYGONS.

he principal symmetric plane figures are the regular polygons and the e.

5. A *regular* polygon is an equilateral polygon of which vertices are equally distant from an interior point called re *of regularity*, or simply *centre*.

regular polygon is often designated by the letter at its centre only.

6. The *radius* of a regular polygon is the line which joins centre to a vertex; the *apothem* is the perpendicular wn from the vertex to a side: two *adjacent* radii are two ii terminating at the extremities of the same side.

is evident that all the radii of a regular polygon are equal to each r.

7. The *centre* angle of a regular polygon is that formed

by two adjacent radii: the triangle formed by two adjacent radii and the subtending side is a *sector* triangle of the polygon.

It follows from the definition of a regular polygon that its sector triangles are isosceles, and that there are as many as there are sides of the polygon.

It is also evident, from what has been seen of the properties of isosceles triangles [8 iv], that the apothem to each side of a regular polygon bisects that side and the centre angle subtended thereby, and conversely, that the bisectrix of each centre angle, or the middle perpendicular to each side, is an apothem.

THEOREM 55.

The centre angles of a regular polygon are equal to one another.

Because a regular polygon is equilateral and its radii are equal to one another, the sector triangles of the polygon are equilateral to each other; hence they are equal to each other [18]: therefore they have the same angle at the apex; that is, the centre angles of the polygon are equal to one another, W. W. T. B. D.

COROLLARY I. *The sector triangles of a regular polygon are equal to each other.*

COROLLARY II. *The apothems of a regular polygon are equal to one another.*

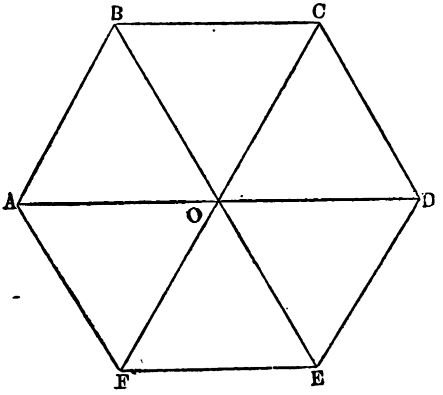
COROLLARY III. *All regular polygons of the same number of sides have equal centre angles.*

COROLLARY IV. *The primary diagonals subtending the alternate angles of an even-sided regular polygon, are the sides of a regular polygon having the same centre and radius, but half the number of sides.*

THEOREM 56.

The prolongation, beyond the centre, of each radius of an even-sided regular polygon is a radius.

Let there be a regular hexagon O, and let O A, O B, O C, O D, O E, ad O F, be its radii.



Because the centre anles A O B, B O C, etc., re equal to one another 55], the angles A O B, B O C, and C O D, are together equal to the angles A O F, F O E, and E O D [iv]; hence the angle A O D is equal to its explement: therefore it is a straight angle [II. 5]; that is, the radius O D is the prolongation of the radius A O. In the same way it could be proved that the radii O E and O F are the prolongations of the radii O B and O C respectively. The same construction and reasoning could be applied to any other even-sided regular polygon; therefore, the prolongation beyond the centre of each radius of an even-sided regular polygon, is a radius, W. W. T. B. D.

COROLLARY I. Conversely, *if the prolongation, beyond the centre, of a radius of any regular polygon be a radius, the polygon is even-sided.*

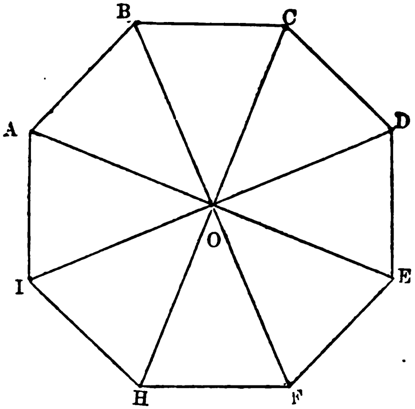
COROLLARY II. *The prolongation, beyond the centre, of each radius of an odd-sided regular polygon, is an apothem,* and conversely.

THEOREM 57.

A regular polygon is equiangular.

Let there be a regular polygon O of any number of sides.

Let the centre be joined to the vertices, A, B, C, etc.; then the sector triangles A O B, B O C, C O D, etc., are equal to one another [55 i]. Because the triangle A O B is isosceles, the angles O A B and O B A are equal to each other [8]; because the angle O B C is equal to its homologous O A B, the angles ABO and OBC are together equal to the angles O A B and O B A [ii]: therefore the angle A B C is supplementary to the centre angle A O B [3]. The same reasoning would prove that the remaining angles BCD, CDE, DEF, etc., are supplementary to the centre angles of the polygon O: but the centre angles are equal to one another [55]; therefore the angles of the polygon O are equal to one another [II. 15 i], W. W. T. B. D.



Corollary I. *Each angle of a regular polygon is supplementary to its centre angle.*

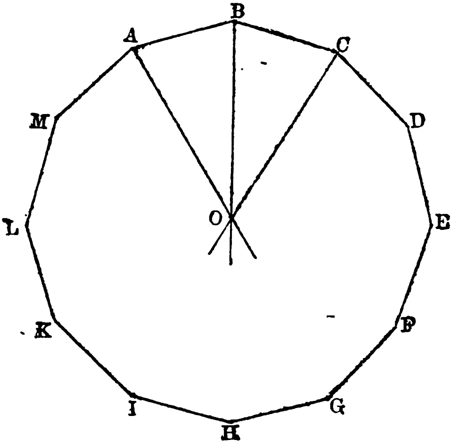
Corollary II. *Each angle of a regular polygon is bisected by the radius terminating at its vertex.*

Theorem 58.

An equilateral polygon which is also equiangular, is regular.

Let there be an equilateral polygon A B C D E F G H I K L M, having its angles A, B, C, D, etc., equal to one another.

Let the bisectrices of the angles A, B and C be drawn, and let O and O′ be the points where the bisectrices of the angles A and C respectively meet the bisectrix of the angle B. Because the angles A, B and C are equal to each other [*Hyp.*], the angles O A B, O B A, ıd O′CB, are equal to each other [vii]; hence the tri- OB and BO′C are isosceles [7]. Because the isosceles A OB and BO′C have equal bases [*Hyp.*], and equal angles se, they are equal to each other [16]; thence BO is equal therefore the ınd O′ are one same point, OB and OC to one an- hat is, the is equally rom the ver- and C. The struction and would prove vertex D and ning vertices ygon, are all listant from O; therefore



on is regular [*Def.* 35], W. W. T. B. D. Inversely *quilateral polygon be not regular, it is not equiangular.*

ARY I. *An equilateral triangle, or a square, is a regular polygon.*

ARY II. *The bisectrices of a regular polygon meet at the centre.*

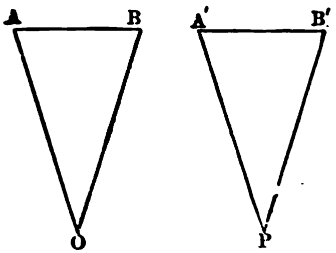
THEOREM 59.

regular polygons of the same number of sides, having dii, are equal to each other.

re be two regular polygons O and P of the same number

OB be a sector triangle of the polygon O and A′PB′ a angle of the polygon P; then the centre angles AOB and

A′PB′ are equal to each other [55 iii]. Because the isosceles triangles A O B and A′PB′ have equal angles at the apex formed by equal sides [*Hyp.*], they are equal to each other [17]; hence, the polygons O and P are composed of the same number of equal triangles similarly placed: therefore they are equal to each other [39], W. W. T. B. D.



Corollary I. *Two regular polygons of the same number of sides, which have one side of the one equal to one side of the other, are equal to each other.*

Corollary II. *Two regular polgyons of the same number of sides are similar to each other.*

Corollary III. *If two regular polygons are equal, or similar, to each other, each sector triangle of the one is equal, or similar, to each sector triangle of the other.*

Scholium. Any sector triangle of each of two equal, or similar, regular polygons, is homologous to any sector triangle of the other.

Theorem 60.

The perimeters of two regular polygons of the same number of sides, are to each other as their sides.

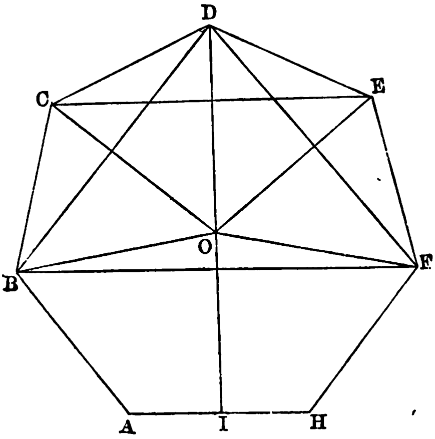
Because two polygons of the same number of sides are similar to each other [59 ii], their perimeters are to each other as their homologous sides [41 i]; but the sides of each of them are equal to one another [*Def.* 35]: therefore the perimeter of the one is to the perimeter of the other, as any one side of the first is to any one side of the second, W. W. T. B. D.

Corollary I. *The perimeters of two regular polygons of the same number of sides, are to each other as their radii.*

THEOREM 61.

A regular polygon is symmetric to each radius produced yond the centre.

Let there be a regular heptagon O, and let O A, O B, O C, etc., be radii.



Let C E be joined, and the radius O D be pro- iced beyond the centre to a point I of the rimeter; then O I is apothem [56 ii]. Be- use D C is equal to D E d the radii O C and O E e equal to each other *Def.* 35], D I is the iddle perpendicular on E [I. 15 ii]: therefore he points C and E are ymmetric to D I [I. 29]. Let B F be joined, and et the point D be join- d to B and F. Be- ause the centre angles B O C, C O D, D O E and E O F are equal to ach other, the angle B O D is equal to the angle D O F [iv]; because he isosceles triangles B O D and D O F have radii of the polygon r their sides, and equal angles at the apex, they are equal to ach other [17]: therefore B D is equal to D F. Because the points and O are equally distant from the extremities of B F, the line D I the middle perpendicular on B F [I. 15 ii]; therefore the points B and are symmetric to D I [I. 29]. The same construction and reasoning ould prove that the points A and H are symmetric to D I; hence e lines D C, C B, B A and A I are the symmetrics of the lines D E, F, F H and H I respectively: therefore the polygon O is symmetric to D I [47 i]. The same construction and reasoning would prove that the polygon O is symmetric to each of the remaining radii produced beyond the centre O; also the same could be applied to any other regular polygon: therefore every regular polygon is symmetric to each of its radii produced beyond the centre, W. W. T. B. D.

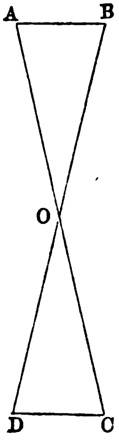
Corollary I. *A regular polygon is symmetric to each apothem produced beyond the centre.*

Corollary II. *A regular polygon has as many axes of symmetry as it has sides or vertices.*

Theorem 62.

An even-sided regular polygon is symmetric to its centre.

Let A O B be a sector of an even-sided regular polygon.



Let A O and B O be produced beyond the centre O to two points C and D of the perimeter of the same polygon, and let C D be joined; then O D and O C are radii, A C and B D diagonals, and D C a side, of the same polygon. Because the isosceles triangles A O B and C O D are equal to each other [55 i], the alternate angles O A B and O C D are equal to each other; thence A B is parallel to C D [II. 21]. But A B and C D are equal to each other [*Def.* 35]; then, as the same construction and reasoning could be applied to the other sides of the polygon O, each side of an even-sided regular polygon is equal and parallel to another side of the same; therefore every even-sided polygon is symmetric to its centre [51 i], W. W. T. B. D.

Corollary I. *An even-sided polygon is symmetric to each of its diameters.*

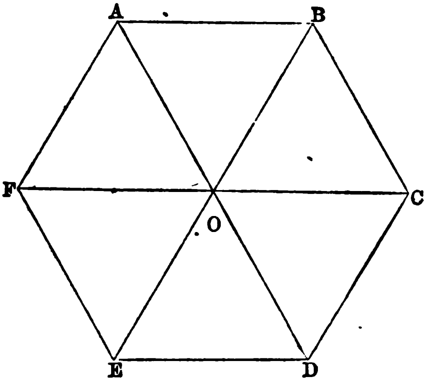
Corollary II. *The perimeters of two even-sided regular polygons of the same number of sides, are to each other as their diameters.*

THEOREM 63.

(*Eucl.* IV. 15.)

Each side of a regular hexagon is equal to the radius.

Let A B C D E F be a regular hexagon, and let O be its centre.



Let the radii O A, O B, tc., be drawn. Because he six centre angles of he hexagon O are together equal to *four* right ngles [II. 17 ii], and hey are all equal to one nother [55], the three ngles A O B, B O C, and O O D are together equal to two right angles [vii]; herefore the angle A O B s equal to *the third part* f two right angles [vii]. Because the six angles of the hexagon O are together equal o *eight* right angles [4], and are equal to one another [57], the hree angles A, B and C are together equal to *four* right angles vii]; thence the partial angles into which the angles A, B and are divided by the radii O A, O B and O C, are together equal to ur right angles. Because the radii O A, O B, O C bisect the ngles A, B and C [57 ii], the six angles O A F, O A B, O B A, O B C, C B and O C D are equal to one another [vii]; hence the three ngles O A F, O A B and O B A are together equal to two right ngles [vii]: therefore each of the angles O A B and O B A is equal *the third part* of two right angles [vii]. Because the angles O B, O A B and O B A are each equal to *the third part* of two ight angles [*Dem.*], they are equal to each other [i] and the triangle O B is equilateral [7 i]; that is, A B is equal to O A: but the exagon O is equilateral [*Hyp.*]; therefore each of its sides is equal the radius O A, which W. T. B. D.

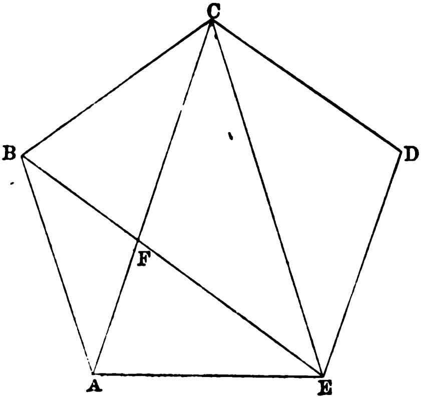
Corollary. *The apothem of a regular triangle is equal to half the radius.*

Theorem 64.

(*Eucl.* IV. 11.)

Each side of a regular pentagon is equal to the mean of a diagonal.

Let A B C D E be a regular pentagon.



Let the diagonals A C, C E and B E be drawn, and let F be the point of section of A C and B E. Because the pentagon A B C D E is equilateral [*Hyp.*], and the angles A, B and D are equal to one another [57], the isosceles triangles A B E, B A C and D C E are equal to each other [17]; thence the diagonals A C, B E and C E are three equal straight lines, and the angles A C B, B A C, A B E and B E A are equal to one another; therefore the triangle A B F is isosceles [7], and F C is equal to F E [iv]. Because the isosceles triangles A F B and E F C have an equal angle at the apex [II. 3], they are similar to each other [24 ii]; hence the angles E C F and F E C are each equal to the angle A B F, or A B E [22]: but the angle A E B is equal to the angle A B E [*Dem.*]; therefore the angle A E C is equal to twice the angle A C E. Because the triangle A C E is isosceles [*Dem.*] and the angle A E C is equal to twice the angle at the apex C, the base is equal to the mean of each of the other sides [15]; that is, A E is equal to the mean of A C: but the sides of the pentagon A B C D E are equal to each other [*Hyp.*]; therefore each side is equal to the mean of A C, which W. T. B. D.

Corollary I. *The diagonals of a regular pentagon are equal to one another.*

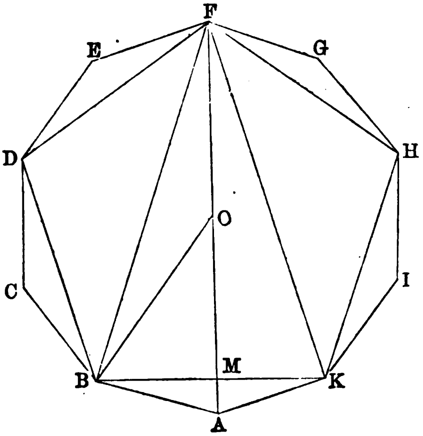
Corollary II. *The angles at the base of the isosceles triangle formed by each side of a regular pentagon, with the adjacent diagonals to the opposite vertex, are each equal to twice the angle at the apex.*

Corollary III. *The diagonals of a regular pentagon cut each other in extreme and mean ratio.*

Corollary IV. *Each side of a regular pentagon is parallel to the diagonal to which it is not adjacent.*

THEOREM 65.

Each side of a regular decagon is equal to the mean of the radius.



Let O F be the radius of a regular decagon A B C D E F G H I K. Let F D, D B, B K, K H and H F be joined; then B D F H K is a regular pentagon of which O F is a radius [55 iv]. Let O F be produced beyond the centre O to a point of the perimeter, through a point M of B K; let also the radius O B be joined: then the prolongation of O F is a radius O A of the decagon [56] and O M is an apothem of the pentagon [56 ii]. Because the angle A O B is exterior to the triangle B O F, it is equal to the sum of the angles O B F and O F B [3 i]: but the triangle B O F is isosceles; then the angles O B F and O F B are equal to each other [8]; therefore the angle A O B is equal to twice the angle O F B. Because the triangle

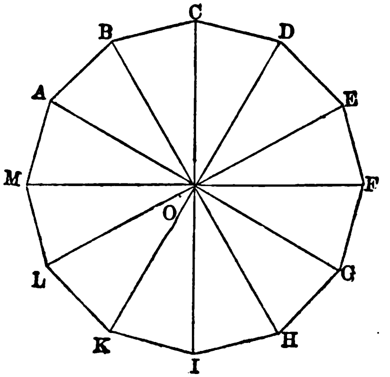
B F K is isosceles [64 i] and F M is a perpendicular to B K [*Dem.*], the line F M is the bisectrix of the angle B F K [8 iv]; that is, the angle B F K is equal to twice the angle O F B: therefore the angle A O B is equal to the angle B F K [i]. Because the isosceles triangles B F K and A O B have an equal angle at the apex, they are similar to each other [24 ii], and the angles at the base of the one, are equal to the angles at the base of the other [22]. Because the angles at the base B K of the triangle B F K are each equal to twice the angle at the apex F [64 ii], the angles at the base A B of the triangle A O B are each equal to twice the angle at the apex O; therefore A B is equal to the mean of O B [15]: but the decagon O is equilateral [*Hyp.*]: therefore each side of the regular decagon O is equal to the mean of the radius O B, which W.T.B.D.

Corollary. *Each bisectrix of the angles at the base of each sector triangle of a regular decagon, divides the radius into extreme and mean ratio.*

Theorem 66.

If a regular polygon be made to revolve upon its centre, until one vertex has taken the place of its adjacent, the polygon is not out of its first place.

Let there be any regular polygon O.



Let O A, O B, O C, etc., be the radii of the polygon O, and let the polygon be conceived to revolve upon its centre O until the radius O B has taken the place of the radius O A; then the vertex B has taken the place of the adjacent vertex A [*Def.* 35]. Because the centre angles A O B and B O C are equal to each other [55], the radius O C has taken the ace of the radius O B, and the vertex C the place of the rtex B; therefore the side B C has taken the place of the side B. The same reasoning would prove that each side of the polygon as taken the place of its adjacent; that is, the perimeter of the olygon has kept the same place on the plane containing it: therefore e polygon O is not out of its first place, W. W. T. B. D. Inversely *If, after revolving upon a point equally distant from three adjacent rtices until one of them has taken the place of its adjacent, a polygon s out of its first place, it is not regular.*

CORROLLARY I. Conversely, *if after revolving upon a point equally disant from three adjacent vertices until one of them has taken the place f its adjacent, a polygon be not out of its first place, it is regular.*

COROLLARY II. *A regular polygon may take as many different posiions relatively to its centre, as it has sides, without being out of its rst place,* and conversely.

SECTION III.

ON THE CIRCLE.

DEFINITIONS.

38. A *curvilinear* plane figure is that which is limited by ne or more curves, or by curved and straight lines. A urve is the *perimeter*, or *periphery*, of the plane figure which t limits.

Among the curvilinear plane figures, Elementary Geometry studies only he circle, and the plane figures which are derived from the circle.

39. A *circle* is the plane figure limited by a circumference f the circle; the centre and radius of the circumference are he *centre* and *radius* of the circle.

It is evident that a circle is symmetric to its diameter [I. 44 ii] and to its entre.

It is also obvious that equal circles have equal circumferences and radii, and conversely that circles which have equal circumferences or equal radii, or the circumferences of which coincide with one another, are equal to one another; whence, if the centres of equal circles coincide with each other, they have the same circumference.

A circle is designated by the letter at its centre, or by the letter at its centre and one or more letters at some points of its circumference, or simply by the latter.

THEOREM 67.

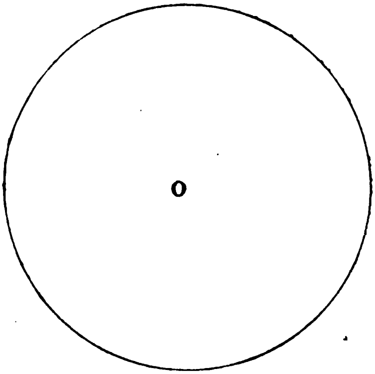
A circle, in revolving upon its centre, does not come out of its first place.

For otherwise, the circumference of the circle would have points unequally distant from the centre, which is impossible [I. *Def.* 23].

COROLLARY. *A circle, in revolving upon its centre, may take an infinite number of different positions without coming out of its first place.*

THEOREM 68.

Every circle is a regular polygon of an infinite number of sides.



Let there be a circle O.

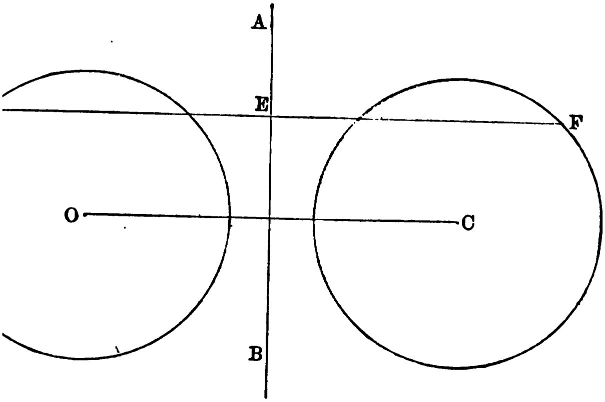
Because the circumference of the circle O is composed of infinitely small straight lines [I. *Def.* 2], the circle O is a polygon the sides of which are infinitely small [*Def.* 2]. Because a point equally distant from any three points of the circumference O is its centre [I. 45 iii], and because a circle ıg upon its centre does not come out of its first place [67], the) is a regular polgyon [66 i]. Now, it is evident that the) may take an infinite number of positions relatively to its without coming out of its first place, for between two po- ıowever little different from each other, it is always possible ɜive a third one; hence the circumference of the circle O osed of an infinite number of sides [66 ii]: therefore the is a regular polygon of an infinite number of sides, W. W. T.

LARY I. *The radius and apothem of a circle are equal to each*

LARY II. *All circles are similar to one another.*

THEOREM 69.

equal circles are symmetric to the middle perpendicular · centre line.



C be the centre line of two equal circles O and C, and let ts middle perpendicular.

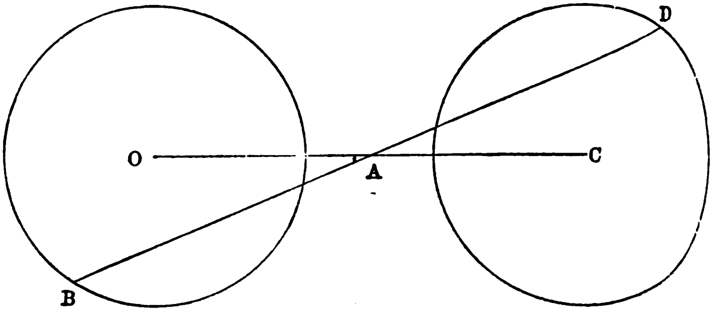
From a point D of the circumference O, let a perpendicular D E be drawn to A B and produced indefinitely. Whether D E be a tangent or a secant to the circumference O, because D E is parallel to O C [I. 16], its continuation beyond the point E will be tangent or secant in a point F to the circumference C [I. 70 ii]; therefore D E is equal to E F [I. 68 iii]. Thus the extremity of every perpendicular drawn from the circumference O to the line A B, and produced beyond that line to a distance equal to itself, is a point of the circumference C; therefore the circles O and C are symmetric to the line A B [*Def.* 28], which W. T. B. D.

COROLLARY I. *Two equal tangent, or secant, circles are symmetric to their intangent, or their common chord.*

COROLLARY II. *Two circles are symmetric to the middle perpendicular on three lines joining their circumferences.*

THEOREM 70.

Two equal circles are symmetric to the middle point of their centre line.



Let O C be the centre line of two equal circles O and C, and A its middle point.

et a point B of the circumference O, be joined to the point A and duced indefinitely; then if B A be a tangent to the circumference its continuation beyond the point A will be a tangent in a point D the circumference C [I. 71 iv]: therefore B A is equal to A D 71]. If B A be a secant to the circumference O, its continuation l be a secant in D to the circumference C [I. 72 iv]; therefore is equal to A D [I. 72 iii]. Thus the extremity of every straight drawn from one of the circumferences O and C to the point A, produced beyond A to a distance equal to itself, is a point of the er circumference; therefore the circles O and C are symmetric he point A [*Def.* 31], which W. T. B. D.

OROLLARY I. *Two equal tangent circles are symmetric to their point ngence.*

OROLLARY II. *If three lines joining the circumferences of two circles t each other, the two circles are symmetric to their intersection.*

THEOREM 71.

he circumferences of two circles are to each other as thei ii.

r two circles are regular polygons of the same number of sides : therefore their circumferences are to each other as their i [60 i].

OROLLARY. *The circumferences of two circles are to each other as diameters.*

SECTION IV.

ON CIRCULAR SEGMENTS AND SECTORS.

DEFINITIONS.

40. A *segment* is the portion of a circle limited by an arc and its chord; the chord is the *base* of the segment. A *semicircle* is a segment equal to half the circle: a segment smaller than a semicircle is a segment *of half the circle*, or a *semicircular* segment; a segment greater than a semicircle is a segment *of the circle*, or a *circular* segment.[1] *Opposite* segments of a circle are limited by the same chord.

Hence, every chord divides the circle into two opposite segments.[2] It is evident that two equal segments have the base and arc of the one, respectively equal to the base and the arc of the other.

A segment is designated by the letters of its base followed by one or more letters of its arc.

41. The *altitude* of a segment is the middle perpendicular on the base from a point of the arc.

It has been shown that the altitude of a segment bisects the arc, and that its prolongation beyond the bases passes through the centre of the circle [I. 44 i].

42. An angle of which the vertex is a point of the arc of a segment, and the sides of which pass through the extremities of the base is *inscribed* in the segment; an angle which may be inscribed in a segment is *inscriptible* therein, and the segment is *capable* of the angle.

---

[1] In The Elements, the term *circular segment* has this restricted meaning only when used in opposition to that of *semicircular segment*; otherwise it means *any* segment.

[2] When, in The Elements, the segment formed by a chord is spoken of, without anything further being said, the segment *limited by the lesser arc* only is meant.

43. A *sector* is the portion of a circle limited by an arc nd the radii terminating at its extremities: the radii are he *sides* of the sector; the centre angle formed by them nd standing upon the arc of the sector, is its *angle*. A ector smaller than a semicircle is a sector *of half the circle*, r a *semicircular* sector; a sector greater than a semicircle s a sector *of the circle*, or a *circular* sector.[1]

It is evident that two equal sectors have the sides, angle and arc of the ne, respectively equal to the sides, angle and arc of the other. It is bvious that a sector the angle of which is straight, is a segment; also that he segment the base of which passes through the centre, is a sector.

A sector is designated by the letters of its angle followed by one or more tters of its arc.

44. The *segment* of a sector is that formed by its arc, he *sector* of a segment is that limited by its arc.

The segment formed by a diameter is evidently identical to its sector. Also, a circular segment is composed of its sector and an isosceles triangle; nd a semicircular sector is composed of its segment and an isosceles triangle.

45. A *quadrant* is a sector the arc of which is the fourth part of the circumference of the circle[2]; a sector the arc of which is the sixth part of the circumference is a *sextant*; a sector of which the arc is the eighth part of the circumference is an *octant*.

THEOREM 72.

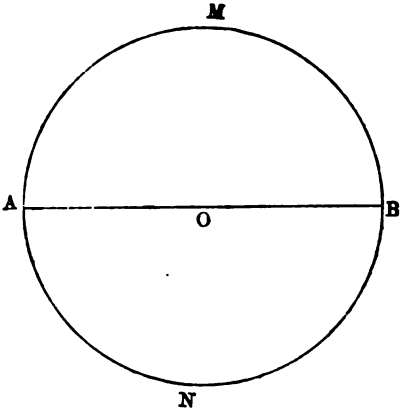
A diameter divides the circle into two equal segments.

---

[1] In the Elements, the term *circular sector* has this restricted meaning only when used in opposition to that of *semicircular sector*; otherwise it means *any* sector.

[2] As the circumference of a circle is sometimes called a circle, so the arc of a quadrant is sometimes called a *quadrant*.

Let A B be a diameter of the circle O.



Let the figure be folded upon A B; then the circumference of the circle will be divided into two equal parts, A M B and A N B, and the two halves will coincide with each other [I. 39]; therefore the segments A B M and A B N are equal to each other, W. W. T. B. D.

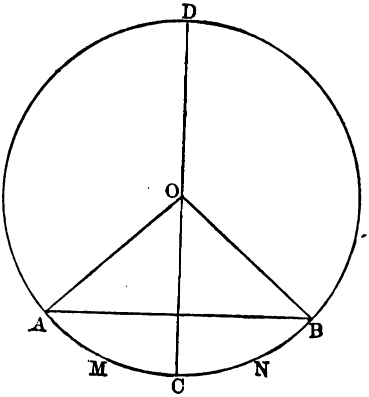
COROLLARY. Conversely, *the base of a semicircle is a diameter.*

Scholium. A diameter divides the circle into two equal sectors.

THEOREM. 73.

The diameter perpendicular on the chord of the arc of a sector, divides the sector into two equal parts.

Let A O B C be a sector; let A B be the chord of its arc, and let C D be a diameter perpendicular on A B, dividing the sector A O B C into two partial sectors, A O C M and B O C N.



Let the circle be folded upon C D; then the semicircle C D A will coincide with the semicircle C D B [72]. Because C D is the middle perpendicular on A B [I. 44], the points A and B are symmetric to O D [I. 29], and will coincide with each other; hence the arc C A will coincide with the arc C B, and the radius O A with the radius O B: that is, the sector A O C M will coincide with the sector B O C N; therefore they are equal to each other, W. W. T. B. D.

Corollary I. *The diameter perpendicular on the chord of the arc of a sector, bisects the angle of the sector.*

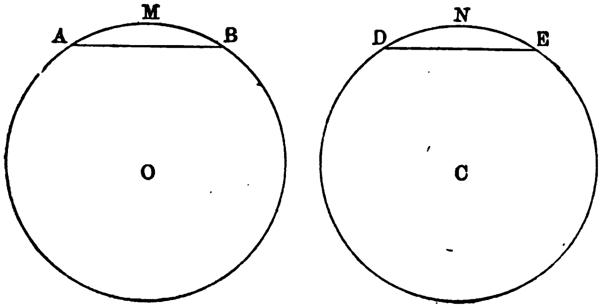
Corollary II. *The diameter perpendicular on the base of a segment bisects the segment.*

Corollary III. *A segment is bisected by its altitude.*

Theorem 74.

Equal segments are parts of equal circles.

Let A B M and D E N be two equal segments.



Let A B M be a portion of the circle O and D E N a portion of the circle C, and let the two segments be applied to each other. Because the segment A B M is equal to the segment D E N, they will coincide with each other [*Gen. Def.* 18], and the arc A M B will coincide with the arc D N E; hence the centres O and C will coincide with each other [I. 46 i], and the circumference O will coincide with the circumference C [I. 58]: therefore the circle O is equal to the circle C, which W. T. B. D. Inversely

Unequal circles cannot have equal segments.

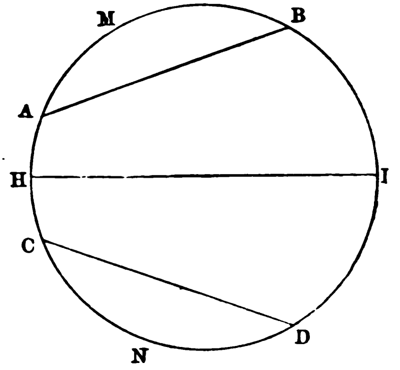
Corollary I. *Equal sectors are parts of equal circles.*

Corollary II. *Equal segments are parts of equal sectors*, and conversely.

Theorem 75.

Two semicircular segments of a circle, which have equal bases, are equal to each other.

Let A B M and C D N be two segments of the circle A B C, having the base A B equal to the base C D.



Let H be the middle point of the arc A C; let H I be a diameter through the point H, and let the circle A B C be folded upon H I: then the semicircle H I M will coincide with the semicircle H I N [72]. Because the arc H A is equal to the arc H C [*Const.*], the point A will coincide with the point C; because the chord A B is equal to the chord C D, the arc A M B is equal to the arc C N D [I. 43]: hence the arc A M B will coincide with the arc C N D, and the chord A B with the chord C D; that is, the segment A B M will coincide with the segment C D N; therefore they are equal to each other, W. W. T. B. D. Inversely

Two unequal segments of a circle have unequal bases, and

Unequal semicircular segments which have equal bases, are not portions of the same circle.

Corollary I. *Two circular segments of a circle, which have equal bases, are equal to each other.*

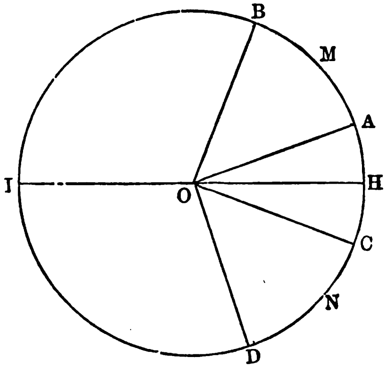
COROLLARY II. *Segments of equal circles which have equal bases, are ual to one another.*

COROLLARY III. *Segments of a circle, or of equal circles, which have ual arcs, are equal to one another.*

THEOREM 76.

Sectors of a circle which have equal angles are equal to each her.

Let A O B M and C O D N be two sectors of the circle O, having e angle A O B equal to the angle C O D.



Let the diameter H I sect the angle A O C, and t the circle O be folded on H I; then the semi- cle H I M will coincide ith the semicircle H I N 2]. Because the angle O A is equal to the angle O C [*Const.*], the radius A will coincide with the dius O C; because the gle A O B is equal to the gle C O D, the radius O B ll coincide with the ra- us O D: then the sector O B M will coincide with the sector C O D N; therefore they are ual to each other, W. W. T. B. D. Inversely

Unequal sectors of a circle have unequal angles, and

Unequal sectors having equal angles are not portions of the same circle.

COROLLARY I. *Sectors of equal circles which have equal angles are ual to one another.*

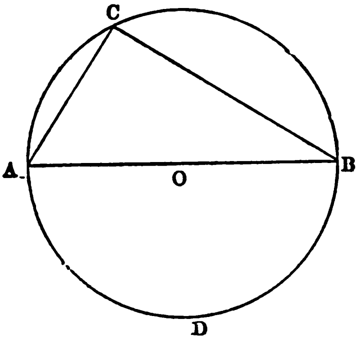
COROLLARY II. *Sectors of a circle, or of equal circles, which have ual arcs, are equal to one another.*

THEOREM 77.

(*Eucl.* III. 31.)

The angle inscribed in a semicircle is right.

Let A C B be an angle inscribed in one half of the circle O.



Let A B D be the other half of the circle O; then the angle A C B is inscribed in the circumference O, and stands upon the arc A D B. Because A B D is a semicircle, the base A B is a diameter [72 i]; then the arc A D B is half the circumference A C D [I. 39]; therefore the inscribed angle A C B is a right angle [II. 33 i], W. W. T. B. D.

COROLLARY I. Conversely, *a segment capable of a right angle, is a semicircle.*

COROLLARY II. *A greater segment is capable of a smaller angle,* and conversely (II. 27 iii).

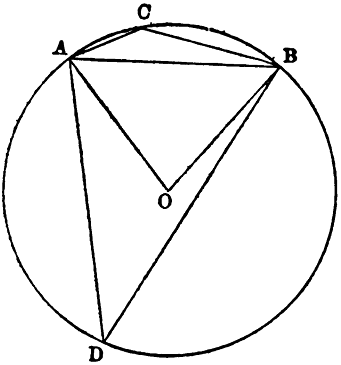
Scholium. A semicircular segment is capable of an obtuse angle, and a circular segment is capable of an acute angle (II. 33 ii).

THEOREM 78.

The angle of a sector is equal to twice the supplement of the angle inscribed in its segment.

Let A O B C be a sector of the circle O; let A B C be its segment, and let A C B be an angle inscribed in that segment.

Let the angle A D B be inscribed in the segment A B D; then the angles A C B and A D B are supplementary to each other [II. 33 iii]. Because the centre angle A O B and the inscribed angle A D B stand upon the same arc, the angle A D B is equal to half the angle A O B [II. 27]; therefore the angle A O B is equal to twice the supplement of the angle A C B, which W. T. B. D.



Corollary I. *The angle inscribed in a segment is equal to half the explement of the angle of its sector.*

Corollary II. *Any segment is capable of the supplement of the angle inscriptible in the opposite segment* (II. 33 iii).

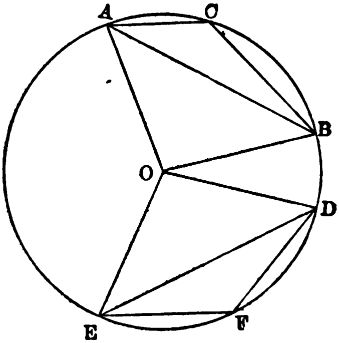
Corollary III. *All the angles inscribed in a segment, or in equal segments, are equal to one another.*

Corollary IV. *If two sectors have equal angles, the angles inscribed in their segments are equal one to another,* and conversely.

Corollary V. *The angle formed on one side of a chord by a tangent to the circle, is inscriptible in the opposite segment formed by that chord* (II. 29 ii).

Theorem 79.

Two segments of a circle which are capable of the same angle, are equal to each other.



Let A B C and D E F be two segments of the circle O, capacapable of the same angle.

Let the points A and B be joined to the centre and to a point C of the arc A C B; let also the points D and F be joined to the centre and to a point F of the arc D F E: then the angles A C B and D F E are inscribed in the segments

A B C and D E F. Because the angles A C B and D F E are equal to each other [*Hyp.*], the angle A O B is equal to the angle D O E [78 iv]; hence the sector A O B C is equal to the sector D O E F [76]: therefore the segments A B C and D E F are equal to each other [74 ii], W. W. T. B. D. Inversely

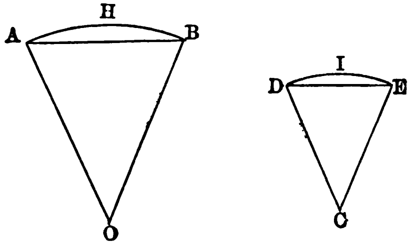
Two unequal segments of a circle are not capable of the same angle.

COROLLARY. *Segments of equal circles which are capable of the same angle, are equal to each other.*

THEOREM 80.

Segments of similar sectors are similar to each other.

Let the sectors A O B H and D C E I be similar to each other, and let A B H and D E I be their segments.



If the two sectors have the same magnitude, they are equal to each other [*Gen. Def.* 18]; therefore their segments are equal [74 ii], and consequently similar [*Gen. Def.* 19], to each other. If the two sectors have not the same magnitude, let them be reduced to the same magnitude; they are then equal to each other [*Gen. Def.* 18], and their segments are equal to each other [74 ii]: but the segments A B H and D E I have also been reduced to the same magnitude [*Post.* I]; therefore they were similar before the reduction, W. W. T. B. D.

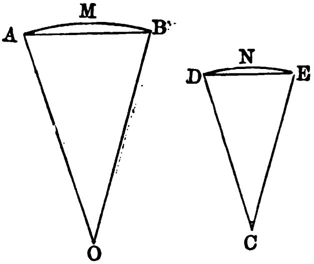
COROLLARY. Conversely, *sectors of similar segments are similar to each other.*

THEOREM 81.

Two similar sectors have equal angles.

Let A O B M and D C E N be two similar sectors.

If the two sectors have the same magnitude, they are equal to each other [*Gen. Def.* 18]; therefore they have equal angles.



If the two sectors have not the same magnitude, let the chords AB and D E be joined; then A O B and D C E are two isosceles triangles. Let the two sectors be reduced to the same magnitude; they will be then equal to each other. Now, let the reduced sectors be applied to each other so that they coincide; then the triangle A O B coincides with the triangle D C E: therefore they are equal to each other. Now, because the equal triangles A O B and D C E have been reduced to the same magnitude [*Post.* I], they were similar to each other before the reduction: therefore the angle A O B is equal to the angle D C E [22], which W. T. B. D. Inversely

Two sectors having unequal angles are not similar to each other.

COROLLARY I. *Similar segments are capable of the same angle* (Eucl. III. Def. 11).

COROLLARY II. *If two sectors be similar to each other, the sides and chord of the one, are proportional to the sides and chord of the other.*

THEOREM 82.

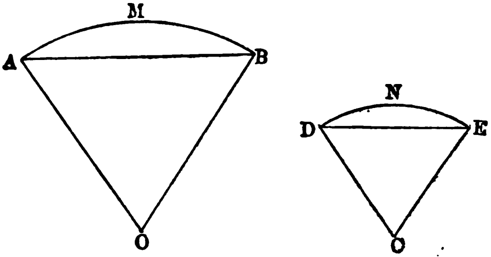
Two sectors having equal angles are similar to each other.

Let A O B M and D C E N be two sectors having the angles A O B and D C E equal to each other.

If the radius O A be equal to the radius C D, the circle O is equal to the circle C; therefore the sectors A O B M and D C E N are equal [76], and consequently similar, to each other.

If O A be not equal to C D, let the chords A B and D E be joined; then the isosceles triangles AOB and DCE are similar to each other [24].

Let the greater sector be reduced until the similar triangles AOB and DCE have the same magnitude [*Post.* I]; these triangles will then be equal to each other [*Gen. Def.* 18]. Now let the reduced sector be applied to the other, so that the equal triangles AOB and D C E coincide with each other; the points A, O and B will then severally coincide with D, C and E. Because the radius O A coincides with the radius C D, they are now equal to each other, and the circumference O is equal to the circumference C [I. 38]; because the centres O and C coincide with each other, the equal circumferences O and C coincide with each other [I. 38 i]; because the points A and B of the circumference O respectively coincide with the points D and E of the circumference C, the arc A M B coincides with the arc D N E, and the sector A O B M coincides with the sector D C E N; hence they are equal to each other: but they have been reduced to the same magnitude [*Const.*]; therefore they were similar to each other before the reduction, W. W. T. B. D. Inversely



Two sectors which are not similar have unequal angles.

COROLLARY I. *Segments capable of the same angle are similar to each other.*

COROLLARY II. *If the sides and chord of a sector be proportional to the sides and chord of another, they are similar to each other* (25 i).

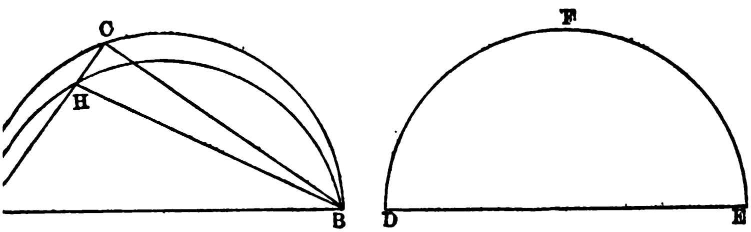
COROLLARY III. *All semicircles are similar to one another.*

THEOREM 83.

(*Eucl.* III. 24.)

Two similar segments which have equal bases are equal to each other.

Let A B C and D E F be two similar segments having the bases A B and D E equal to each other.



Let the two segments be applied to each other, so that the base D E coincide with the base A B, and let there be, through the point A, a straight line A C cutting the arc A C B in C and the arc D F E in H; let also C B and H B be joined: then the angle A C B is inscribed in the segment A B C, and the angle A H B in the segment D E F, or A B H. Because the two segments A B C and A B H are similar to each other [*Hyp.*], the corresponding angles A H B and A C B are equal to each other [81 i]; therefore the lines C B and H B are parallel to each other [II. 20 i]. Because the parallels C B and H B have the point B in common, they coincide with each other [I. 17 iv]; that is, they are one and the same line: hence the points H and C are one and the same point; therefore the arcs A C B and A H B coincide with each other [I. 46 iii]. Because the segments A B C and A B H coincide in all their parts, they are equal to each other: but the segment A B H is no other than the segment D E F [*Const.*]; therefore the segments A B C and D E F are equal to each other, W. W. T. B. D. Inversely

Two unequal segments having equal bases are not similar to each other.

SECTION V.

ON INSCRIBED PLANE FIGURES.

DEFINITIONS.

46. A plane figure *inscribed* in another has its angular points on the perimeter of the latter; the second is *circumscribed* to the first. A plane figure which can be inscribed in another, is *inscriptible* in the latter; the second is *circumscriptible* to the first.

47. A rectilinear plane figure *circumscribed* to a circle has all its sides tangent to the circumference; the circle is then *inscribed* in the rectilinear plane figure.

48. A circle *exscribed* to one side of a rectilinear plane figure is tangent to that side and to the prolongations of the adjacent sides.

THEOREM 84.

(*Eucl.* IV. 5.)

Every triangle is inscriptible in a circle.

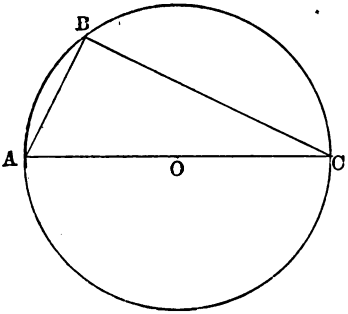
For a circumference may be drawn through the three angular points of any triangle [I. 45 i].

COROLLARY I. *If two rectilinear plane figures be equal to each other, the circles circumscribed to them are equal to each other.*

THEOREM 85.

The centre of the circle circumscribed to a right-angled triangle, is the middle point of the hypothenuse.

Let A C be the hypothenuse of a right-angled triangle A B C inscribed in the circle O.



Because the angle A B C is right, the segment A C B is a semicircle [77 i]: therefore A C is a diameter [72 i]; that is, the centre O is the middle point of the hypothenuse A C, which W. T. B. D. Inversely

If the centre of the circle circumscribed to a triangle be not on one of its sides, the triangle is not right-angled.

Corollary I. Conversely, *if the centre of a circle circumscribed to a triangle is on one of its sides, the triangle is right-angled in the opposite vertex.*

Corollary II. *The centre of the circle circumscribed to an acute angled triangle, is within the triangle,* and conversely.

Corollary III. *The centre of the circle circumscribed to an obtuse-angled triangle, is without the triangle,* and conversely.

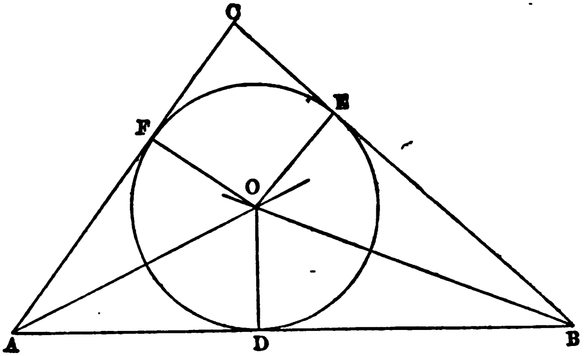
Theorem 86.

(*Eucl.* IV. 4.)

Every triangle is circumscriptible to a circle.

Let there be a triangle A B C.

Let A O and B O be the bisectrices of the angles A and B; from the intersection O [13], let O D, O E and O F be drawn respectively perpendicular to the sides A B, B C and A C. Because A O is the bisectrix of the angle A, the line O F is equal to O D [II. 9]; because B O is the bisectrix of the angle B, the line O E is equal to O D: therefore O F is equal to O E [i]; thus, the circumference drawn from the centre O, with the radius O D, will pass through the points E and F [I. *Def.* 27]. Because the sides of the triangle A B C are perpendicular to the radii O D, O E and O F [*Const.*], they are tangent to the circumference O [I. 51]; therefore the triangle A B C is circumscribed to the circle O [*Def.* 47], which W. T. B. D.



Corollary I. *The circles inscribed in equal triangles are equal to each other.*

Corollary II. *The diameter of the circle inscribed in a right-angled triangle is equal to the difference between the hypothenuse and the sum of the other sides.*

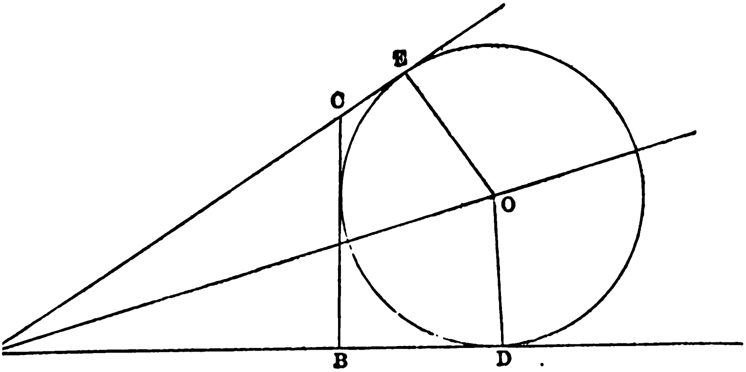
Corollary III. *The sum of each side of a triangle and the remote portion of another, determined by the point of tangence with the inscribed circle, are together equal to half the perimeter.*

Theorem 87.

The centre of the circle exscribed to any side of a triangle is on the bisectrix of the opposite angle.

Let the circle O be exscribed to the side B C of the triangle A B C.

Let D and E be the points of tangence of the circle O with the prolongations of the sides A B and A C; let O D and O E be joined, and let A O be a straight line through the vertex A and the centre O: then O D is equal to O E [I. *Def.* 23]. Because O D is perpendicular to A D [I. 55], and O E perpendicular to A E, the point O is equally distant from the lines A D and A E [I. 11]; therefore the centre O is on the bisectrix of the angle A [II. 11], which W. T. B. D.



Corollary I. *The centre of the circle exscribed to any side of a triangle, that of the inscribed circle and the opposite vertex, are points of the same straight line.*

Corollary II. *The diameter of the circle exscribed to the hypothenuse of a right-angled triangle, is equal to the perimeter of the triangle.*

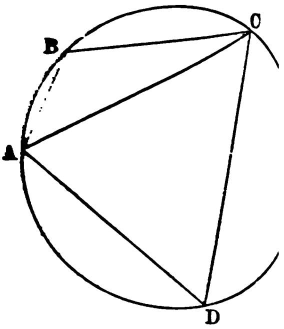
Corollary III. *The distances from the vertex of each angle of a triangle, to the points of tangence of its sides with the circle exscribed to the opposite side, are each equal to half the perimeter.*

Corollary IV. *The distance from the point of tangence of each side of a triangle with the inscribed circle, to one extremity, is equal to the distance from the other extremity to its point of tangence with the exscribed circle.*

THEOREM 88.

The opposite angles of a quadrangle inscribed in a circ are supplementary to each other.

Let the quadrangle A B C D be inscribed in the circle A B D.



Let A C be joined; then the angles A B C and A D C are inscribed in the opposite segments A C B and A C D: therefore they are supplementary to each other [78 ii]. The same construction and reasoning would prove that the angle B A D is supplementary to the angle B C D: therefore the opposite angles of the quadrangle A B C D are supplementary to each other, W. W. T. B. D. Inversely

If two opposite angles of a quadrangle be not supplementary to other, the quadrangle is not inscriptible in a circle.

COROLLARY. *Each exterior angle of a quadrangle inscribed in a c is equal to the opposite interior angle.*

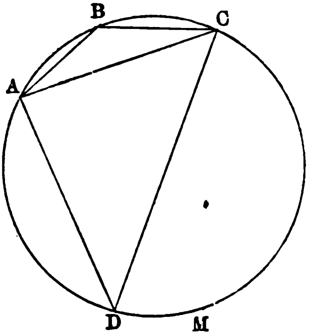
THEOREM 89.

Conversely, *if two opposite angles of a quadrangle be plementary, the quadrangle is inscriptible in a circle.*

Let A B C D be a quadrangle having the angles B and D s mentary to each other.

Let A C be joined, and let the triangle A B C be inscribed i circle A B M; then the segment A C M is capable of the s ment of the angle A B C [78 ii]: therefore the vertex D is o arc A M C, which W. T. B. D.

COROLLARY I. *Every rectangle, or isosceles trapezium, is inscriptible in a circle.*



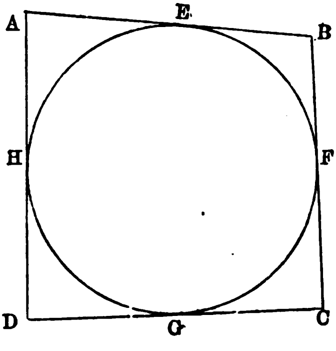
COROLLARY II. *If the portions into which the diagonals of a quadrangle cut each other, be reciprocally proportional, the quadrangle is inscriptible in a circle.*

COROLLARY III. *The centres of the circle exscribed to any side of a triangle, the centre of the inscribed circle and the extremities of the side, are the vertices of a quadrangle inscriptible to a circle.*

Scholium. A square is inscriptible in a circle.

THEOREM 90.

Two opposite sides of a quadrangle circumscribed to a circle are together equal to the two others.



Let A B C D be a quadrangle circumscribed to the circle E F H; and let E, F, G and H be the points of tangence of the sides A B, B C, C D and A D respectively.

Because the sides of the quadrangle A C are tangent to the circle E F H, the lines A E, B E, C G and D G are respectively equal to A H, B F, C F and D H [I. 57]; therefore A E, B E, C G and D G are together equal to A H, B F, C F and D H [iv]: that is, the sides A B and

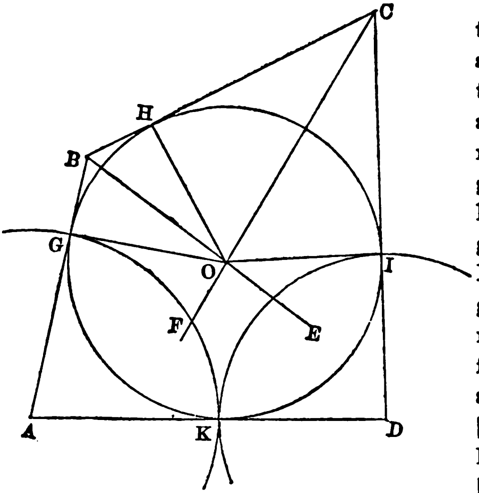
D C are together equal to the sides A D and B C, which W. T. B. D. Inversely

If two opposite sides of a quadrangle be not together equal to the two others, the quadrangle is not circumscriptible to a circle.

THEOREM 91.

Conversely, *if two opposite sides of a convex quadrangle be together equal to the others, the quadrangle is circumscriptible to the circle.*

Let A B C D be a convex quadrangle having the opposite sides A B and D C equal together to the two others A D and B C.



Let B E and C F be the bisectrices of the angles B and C. Because the angles A, B, C and D are together equal to four right angles [47], the angles B and C are together less than four right angles; hence their halves E B C and F C B are together less than two right angles [vii]: therefore the lines B E and C F are oblique to each other [II. 20 ii] and, infinitely produced, will meet [I. 18 i].

Let O be the intersection of the lines B E and C F; let O G, O H and O I be perpendicular to the sides A B, B C and C D respectively. Because the point O is on the bisectrix of the angle B, the line O G is equal to O H [II. 10]; because the same point is on the bisectrix of the angle C, the line O I is equal to O H; therefore O G is equal to O I [i].

Let a circumference be drawn from the centre O with O G as radius; then it will pass through the points H and I [I. *Def.* 3]. Because the sides B A, B C and C D are perpendicular at the extremities of the radii O G, O H and O I respectively, they are tangent to the circle O [I. 54]; hence B G is equal to B H [I. 57], and C I is equal to C H; therefore B G and C I are together equal to B H and H C, or B C [iv]: but A B and D C are together equal to B C and A D [*Hyp.*]; therefore A G and D I are together equal to A D [iv].

From the centre A let a circumference be drawn through the point G, and from the centre D let a circumference be drawn through the point I; then, because their centre line is equal to the sum of their radii [*Dem.*], these two circumferences will be externally tangent to each other in a point K [I. 62]. Because O G and O I are perpendicular at the extremities of the radii A G and D I respectively, they are tangent to the circumferences A and D respectively: because O G and O I are equal to each other [*Dem.*], the circumference O is tangent to A D in the point K [I. 67 ii]: therefore the quadrangle A B C D is circumscribed to the circumference O, which W. T. B. D.

COROLLARY. *Every lozenge is circumscriptible to a circle.*

Scholium. A square is circumscriptible to the circle.

THEOREM 92.

Every regular polygon is inscriptible in a circle.

Because the vertices of any regular polygon are equally distant from the centre of regularity, the circumference drawn from the latter point as a centre, through one of the vertices, will pass through all the others; therefore every regular polygon is inscriptible in a circle, W. W. T. B. D. Inversely

Any polygon which is not inscriptible in a circle is not regular.

Corollary I. *The centre of the circle circumscribed to a regular polygon, is the centre of regularity of the inscribed polygon.*

Corollary II. *The radius of the circle circumscribed to a regular polygon, is equal to the radius of the inscribed polygon.*

Corollary III. *The points of division of a circumference divided into more than two equal parts, are the vertices of a regular polygon.*

Corollary IV. *If a regular polygon be inscribed in a circle, the extremities and the middle points of the arcs subtended by the sides, are the vertices of a regular polygon of double the number of sides.*

Scholium. The sides of a regular polygon are equally distant from the centre (55 ii).

Theorem 93.

Every regular polygon is circumscriptible to a circle.

Because the apothems of a regular polygon are equal to one another [55 ii], the circumference drawn from the centre of regularity with one of them as a radius, will be tangent to all the sides of the polygon [I. 54]; therefore every regular polygon is circumscriptible to a circle, W. W. T. B. D. Inversely

Any polygon which is not circumscriptible to a circle is not regular.

Corollary I. *The centre of the circle inscribed in a regular polygon is the centre of regularity of the circumscribed polygon.*

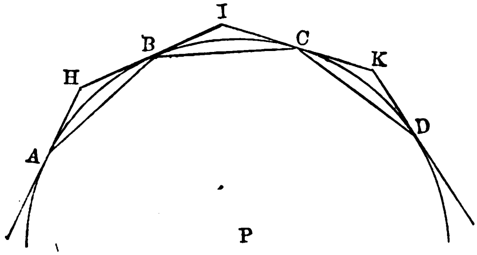
Corollary II. *The radius of the circle inscribed in a regular polygon, is equal to the apothem of the circumscribed polygon.*

Corollary III. *Two adjacent sides of a regular polygon, are symmetric to the radius terminating at their common extremity* (I. 57 iv).

Scholium. The circle circumscribed to a polygon and the circle inscribed therein, are concentric.

Theorem 94.

If a circumference be divided into more than two equal parts, the intersections of the straight lines tangent thereto *through the points of division, are the vertices of a regular polygon.*



Let there be a circumference divided into more than two equal parts at the points A, B, C, D, etc.; let there be, through these points, tangents to the circumference, which meet in the points H, I, K, etc., being the vertices of a polygon P.

Let A B, B C and C D be joined; then the lines A B, B C and C D are equal to one another [I. 42]. Because the circumscribed angles A H B and B I C have equal convex arcs, they are equal to each other [II. 30 iii]: likewise the angle C K D and the remaining angles thus formed, are equal to the angle A H B; therefore the polygon P is equiangular. Because A H is equal to H B [I. 57], B I equal to I C, and C K equal to K D, the triangles A H B, B I C and C K D are isosceles. Because the isosceles triangles A H B, B I C and C K D have equal bases and equal angles at the apex [*Dem.*], they are equal to each other [16 iv]; thence the lines A H, H B, B I, I C, C K and K D are equal to each other: therefore H I is equal to I K [iv]. The same construction and reasoning would prove that the remaining sides of the polygon P are each equal to H I; then the polygon P is equilateral: but it is also equiangular [*Dem.*]; therefore the polygon P is regular [58], W. W. T. B. D.

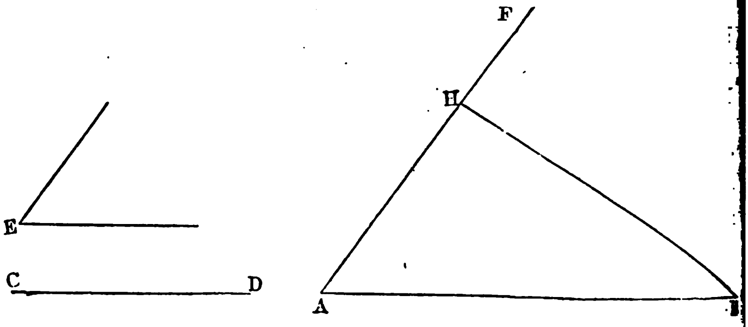
Corollary. *The tangents through the extremities of two perpendicular diameters, form a square.*

PROBLEMS.

PROBLEM 1.

Given two sides of a triangle and the angle which the form, to construct the triangle.

Let A B and C D be the given sides, and let E be the given angle



At the point A draw A F making with A B an angle B A F equal to the given angle E [II. *Prob.* 1]; take A H equal to C D, and join B H: the triangle A B H is the required triangle.

For the side A H is equal to C D [*Const.*], and the angle B A H, formed by the sides A B and A H, is equal to the given angle E [*Const.*], which was to be done.

PROBLEM 2. *Given one side of a triangle and the two adjacent angles, to construct the triangle.*

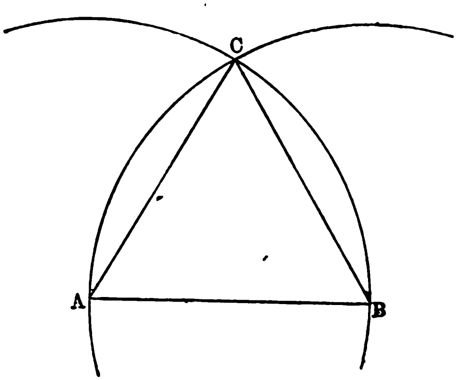
PROBLEM 3.

(*Eucl.* I. 1.)

Upon a given straight line to construct an equilateral triangle.

Let A B be the given straight line.

From the centres A and B, with a radius equal to A B, describe



two circumferences cutting each other in a point C, and join C A and C B: the triangle A B C is the required equilateral triangle.

Because A C and B C are radii of the circumferences A and B, they are equal to A B [*Const.*]; hence they are equal to each other [i]: therefore the triangle A B C is an equilateral triangle having A B for one of its sides, W. W. T. B. D.

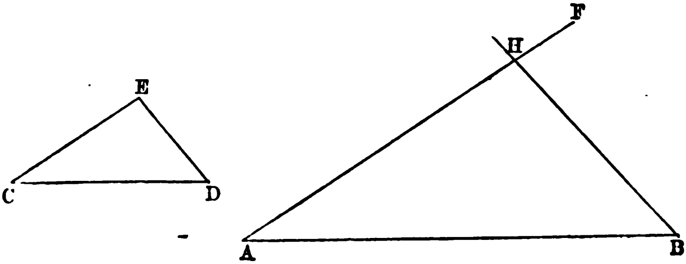
PROBLEM 4. *Given the base and an adjacent side of an isosceles triangle, to construct the triangle.*

PROBLEM 5. *Given the three sides of a scalene triangle, to construct the triangle.*

PROBLEM 6.

Upon a given straight line, to construct a triangle similar to a given triangle.

Let A B be the given straight line and C D E the given triangle.



At the point A, draw A F, making with A B an angle B A F equal to the angle C [II. *Prob.* 1]; and at the point B draw B H, making with A B on the same side, an angle A B H equal to the angle D, and cutting A F in a point H: the triangle A B H is the required triangle.

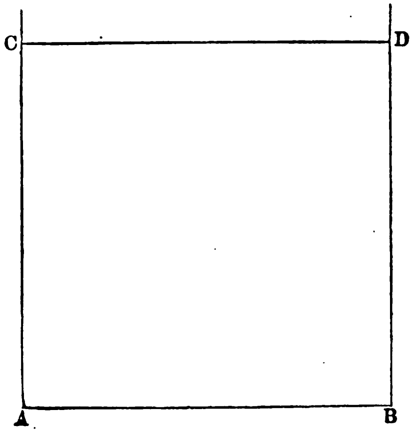
For the angles A and B are respectively equal to the angles C and D [*Const.*]; therefore the triangle A B H is similar to the triangle C D E [23], which W. T. B. D.

PROBLEM 7. *Upon a given straight line, to construct a polygon similar to a given polygon.*

PROBLEM 8.

Upon a given straight line, to construct a square.

Let A B be the given straight line.



Through the points A and B, draw A C and B D perpendicular to A B; take A C and B D both equal to A B and join C D: the quadrangle A B C D is the required square.

Because A C and B D are each equal to A B [*Const.*], they are equal to each other [i]; because the line C D has two of its points, C and D equally distant from A B, it is parallel to A B [I.20]; hence A C and B D are perpendicular to C D [I. 17] and C D is equal to A B [I. 19 i]: that is, the lines A B, A C, B D and C D are all equal to one another, and the angles A, B, C, and D are all right angles; therefore the quadrangle A B C D is a square constructed *upon* A B [*Def.* 18], which W. T. B. D.

PROBLEM 9. *Given two adjacent sides of a rectangle, to construct the rectangle.*

PROBLEM 10. *Given two adjacent sides of a parallelogram and the angle which they form, to construct the parallelogram.*

PROBLEM 11.

To inscribe a given triangle in a circle.

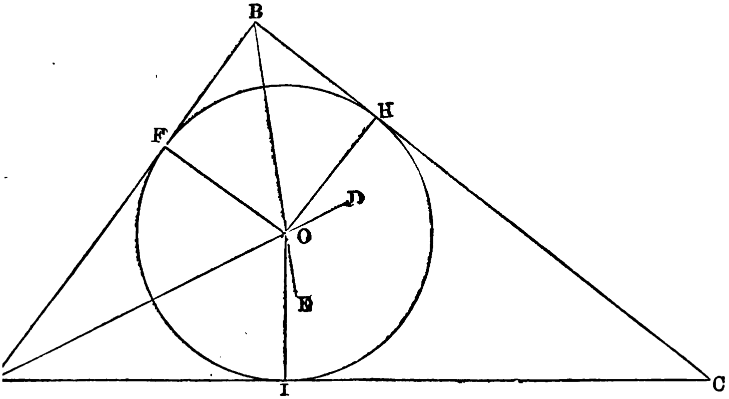
Draw a circumference through its vertices [I. *Probl.* 13]; the given triangle will be inscribed in a circle, for its angular points will be on the circumference.

PROBLEM 12. *To inscribe a given regular polygon in a circle.*

PROBLEM 13.

To inscribe a circle in a given triangle.

Let A B C be the given triangle.



Draw the bisectrices A D and B E of the angles A and B [II. *Prob.* 5]; from the intersection O, draw a perpendicular O F on the side AB

[I. *Probl.* 9]; and, from the centre O, describe a circumference through the point F: the circle O is inscribed in the given triangle.

Draw O H and O I perpendicular to the sides B C and A C respectively. Because the point O belongs to the bisectrix of the angle A, the lines O F and O I are equal to each other [II. 10]; because the same point O belongs to the bisectrix of the angle B, the lines O F and O H are equal to each other: thence the circumference O will pass through the points H and I. Because the sides A B, B C, and A C are perpendicular to the radii O F, O H and O I respectively, they are tangent to the circumference O [I. 54]; therefore the circle O is inscribed in the given triangle A B C, which W. T. B. D.

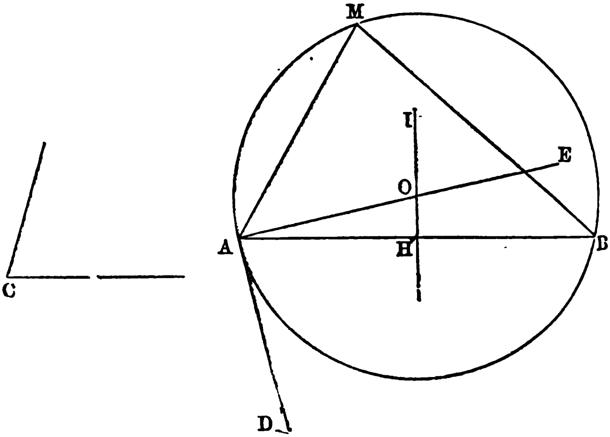
PROBLEM 14. *To inscribe a circle in a given regular polygon.*

PROBLEM 15.

(*Eucl. III.* 33.)

Upon a given straight line, to describe a segment capable of a given angle.

Let A B be the given straight line, and C the given angle.



Through the point A draw A D, making with A B an angle B A D equal to the given angle [II. *Probl.* 1]; draw A E perpendicular on

D [I. *Probl.* 5], and H I middle perpendicular on A B [I. *Probl.* 6], tersecting in a point O; from the centre O, draw a circumference MB through the point A: the segment A B M is the required egment.

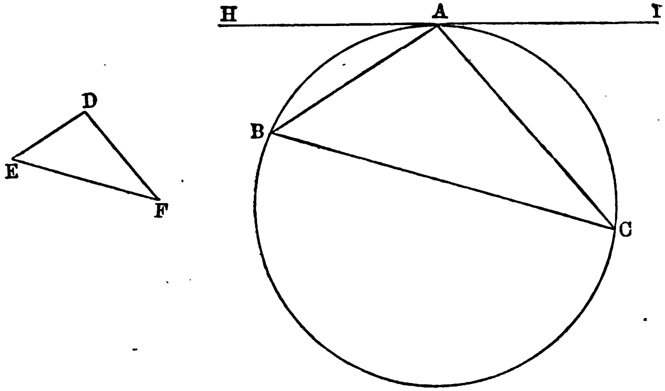
Join a point M of the arc A M B to the points A and B. Because the point O belongs to the middle perpendicular on A B [*Const.*], it is equally distant from the points A and B [I. 14]; hence the circumference O passes through the point B. Because A D is a tangent to the circle O [I. 54], the angle A M B, inscribed in the segment A B M, is equal to the angle B A D [II. 29 ii]; thence the segment A B M is capable of the angle B A D [*Def.* 42]: but the angle B A D is equal to the given angle C [*Const.*]; therefore the segment A B M is capable of the given angle C, which W. T. B. D.

PROBLEM 16. *By a straight line drawn through a point given in the circumference of a given circle, to cut off a segment capable of a given angle.* (Eucl. III. 34.)

PROBLEM 17.

(*Eucl. IV.* 2.)

In a given circle, to inscribe a triangle similar to a given triangle.



Let A B C be the given circle and D E F the given triangle.

Through a point A of the circumference A B C, draw a tangent H I [I. *Probl.* 20]; draw the chord A B making an angle H A B equal to the angle F, and the chord A C making an angle I A C equal to the angle E [II. *Probl.* 1]; and join B C: the triangle A B C is the required triangle.

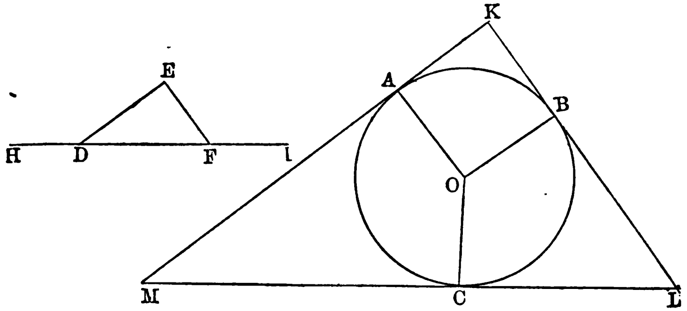
Because H I is a tangent to the circle A B C [*Const.*], the inscribed angle C is equal to the angle H A B [II. 29 ii] and the inscribed angle B is equal to the angle I A C; thence the angles C and B are respectively equal to the angles F and E [i]; therefore the triangle A B C, inscribed in the circle A B C [*Def.* 46], is similar to the given triangle D E F [23], which W. T. B. D.

PROBLEM 18.

(*Eucl. IV.* 3.)

To a given circle, to circumscribe a triangle similar to a given triangle.

Let A B C be the given circle, and let D E F be the given angle.



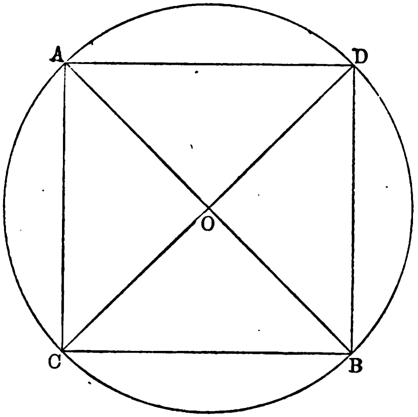
Produce D F in H and I and draw a radius O A; make the centre angle A O C equal to the angle E D H and the centre angle C O B equal to the angle E F I [II. *Probl.* 1], and, through the points A, B

ıd C, draw tangents to the circle O [I. *Probl.* 20] meeting in K, L
ıd M: the triangle K L M is the required triangle.
Because the angles M and L are circumscribed to the circle O
onst.], they are supplementary to the centre angles A O C and B O C
I. 30]: but the angles A O C and B O C are equal to the angles
D H and E F I respectively [*Const.*]; then the angles M and L are
ual to the angles E D F and E F D, each to each [II. 15 i]; there-
re the triangle K L M, circumscribed to the given circle A B C
ef. 47], is similar to the given triangle D E F [23], which W. T.
. D.

PROBLEM 19.

To inscribe a square in a given circle.

Let O be the given circle.
Through the centre O
aw the diameters A B
d C D perpendicular to
ch other, and join A D,
B, B C and A C: the
adrangle A D B C is
e required square.
Because the four centre
gles at the point O are
ht angles [II. *Def.* 11],
y are equal to one
other [II. 15]; hence
arcs A D, D B, B C
d A C are all equal [II.
]: therefore the chords
D, D B, B C and A C are equal to one another [I. 42]. Because the
gles C A D, A D B, D B C and B C A are inscribed in semi-
cles [*Const.*], they are equal to each other [77]; therefore the quad-
ıgle A D B C inscribed in the circle O, is a square, W. W. T. B. D.



PROBLEM 20. *To inscribe a regular octagon in a given circle.*

PROBLEM 21. *To inscribe a regular hexagon, or an equilateral triangle, or a regular dodecagon, in a given circle.*

PROBLEM 22. *To inscribe a regular decagon, or pentagon, in a given circle.*

PROBLEM 23. *To inscribe a regular pentedecagon in a given circle.*

PROBLEM 24.

To circumscribe a square to a given circle.

Inscribe a square in the given circle, and, through each vertex, draw a tangent to the circle: the quadrangle thus formed is a square circumscribed to the circle.

For, it is regular [94], and its sides are tangent to the given circle.

PROBLEM 25. *To circumscribe an equilateral triangle, or a regular hexagon, octagon, decagon, dodecagon, or pentedecagon, to a given circle.*

BOOK IV.

ON AREAS.

BOOK IV.

ON AREAS.

SECTION I.

ON EQUIVALENCE GENERALLY.

DEFINITIONS.

1. When two parallels contain opposite sides of a parallelogram, or the base and apex of a triangle, the parallelogram, or the triangle, is said to be *between* these parallels.

Parallelograms or triangles which are between the same parallels have obviously equal altitudes.

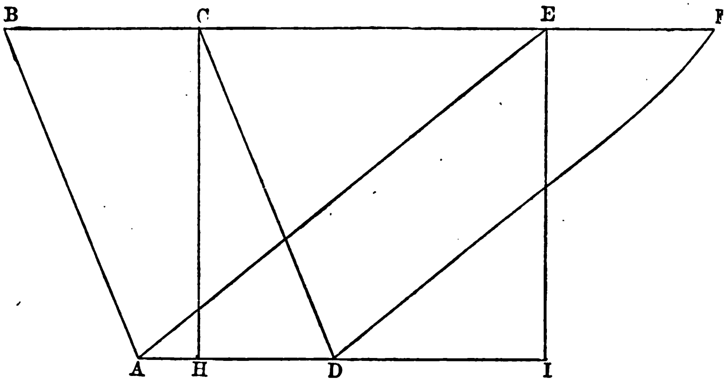
2. Of the four partial parallelograms into which a parallelogram is divided, by parallels drawn to two adjacent sides, through a point of one of the diagonals, those about the diagonal are said to be *completed* by the others; the latter are called the *complements* of the former.

THEOREM 1.

Two parallelograms having equal bases and altitudes, are equivalent to each other.

Let there be two parallelograms A B C D and A E F D, having equal bases and altitudes.

Let these parallelograms be applied to each other so that their equal bases coincide with each other in A D, and let C E be joined; let also C H and E I be perpendiculars from the vertices C and E to the common base A D and its prolongation: then C H is the altitude of the parallelogram A C [III. *Def.* 20] and E I is the altitude of the parallelogram A F. Because the lines C H and E I are equal to each other [*Hyp.*], C E is parallel to A D [I. 20]; but B C and E F are also parallel to A D [*Hyp.*]: therefore B C E F is a straight line parallel to A D [I. 17 iv]. Now, the quadrangle A B F D is composed of the parallelogram A C and the triangle D C F, but it is also composed of the parallelogram A F and the triangle A B E. Because the sides of the angles B A E and C D F are parallel and have the same direction, the angle B A E is equal to the angle C D F [II. 23]; because also A B is equal to C D, and A E to D F [III. 29], the triangles D C F and A B E are equal to each other [III. 17]: thence if either of these two triangles be taken from the quadrangle A B F D, the remainders will be equal in magnitude [ii]; therefore the parallelograms A C and A F are equivalent to each other, W. W. T. B. D.



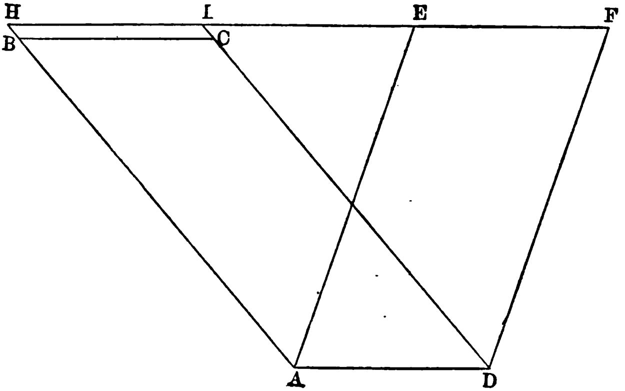
COROLLARY. *If two parallelograms having the same base or equal bases, be between the same parallels, they are equivalent to each other.* (Eucl. I. 35 and 36.)

Scholium. The perimeters of two equivalent parallelograms having equal bases may differ infinitely.

THEOREM 2.

f two equivalent parallelograms have equal bases, they e equal altitudes.

et A B C D and A E F D be two parallelograms having equal es.



Let these parallelograms be applied to each other so that the bases ncide with each other in A D, and let E F be produced indefi- ely; then the prolongation of E F will cut A B and D C, or their longations, in some points H and I. Because the parallelograms A I d A F are between the parallels A D and H F, they are equivalent to h other [1 i]; but A C is equivalent to A F [*Hyp.*], then A C and A I equivalent to each other [i]: therefore the parallelogram B C I H no magnitude; that is, the lines B C and H I are one and the ne line. Because the lines B C and E F are portions of the same allel to A D, they are equally distant from A D [I. 19]; therefore two parallelograms A C and A F have equal altitudes, W. W. T. D. Inversely

f two equivalent parallelograms have unequal altitudes, they have qual bases.

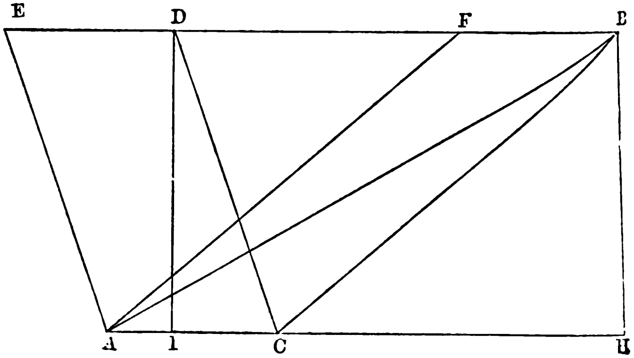
OROLLARY I. *If two equivalent parallelograms have the same base* *or equal bases, on a straight line and on the same side thereof, they are between the same parallels.*

Corollary II. *If two equivalent parallelograms have equal altitudes, they have equal bases.*

Theorem 3.

A triangle is equivalent to half the parallelogram, the base and altitude of which are equal to those of the triangle.

Let there be a triangle A B C and a parallelogram A E D C, having equal bases and altitudes.



Let the triangle and the parallelogram be applied to each other so that their bases coincide with each other in A C, and let B D be joined; let also B H and D I be perpendiculars from the vertices B and D to the common base A C and its prolongation: then B H is the altitude of the triangle A B C, and D I that of the parallelogram A D. Because the lines B H and D I are equal to each other [*Hyp.*], B D is parallel to A C [I. 20]; but E D is also parallel to A C [*Hyp.*]: therefore E D B is a straight line parallel to A C [I. 17 iv]. Now, let A F be a parallel to C B; then A F B C is a parallelogram. Because the parallelograms A E D C and A F B C are between the ɘls A H and E B, they are equivalent to each other [1 i]; thence, ıalves are equivalent to each other [vii]: but the triangle A B C 'the parallelogram A F B C [III. 28]; therefore it is equivalent the parallelogram A E D C [i], which W. T. B. D.

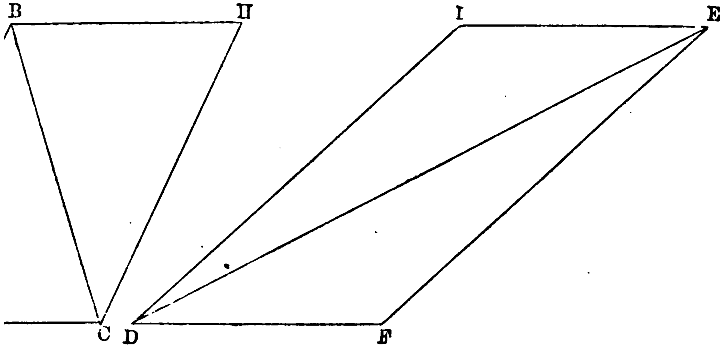
OLLARY I. *If a triangle and a parallelogram having the same r equal bases, be between the same parallels, the triangle is lent to half the parallelogram.* (Eucl. I. 41.)

OLLARY II. *Every triangle is equivalent to the parallelogram of the base and half the altitude, or the altitude and half the base, pectively equal to the base and altitude of the triangle.*

THEOREM 4.

o triangles having equal bases and altitudes are equiva-) each other.

there be two triangles, A B C and D E F, having equal bases titudes.



B H be a parallel to A C and let C H be a parallel to A B; let I be a parallel to F D and let D I be a parallel to F E: then C and D I E F are two parallelograms having the same es as the triangles A B C and D E F [III. *Def.* 20]. Because allelograms A H and F I have equal bases and altitudes, they ivalent to each other [1]; thence their halves are equivalent to each other [vii]: but the triangle A B C is half the parallelogram A H, and the triangle D E F is half the parallelogram F I [III. 28 i]; therefore the triangles A B C and D E F are equivalent to each other [i], W. W. T. B. D.

Corollary I. *If two triangles having the same base or equal bases, be between the same parallels, they are equivalent to each other.* (Eucl. I. 37 and 38.)

Corollary II. *If two triangles have two sides of the one equal to two sides of the other, each to each, and if the angles formed by these sides be supplementary, these triangles are equivalent to each other.*

Corollary III. *If the points which divide the base of a triangle into equal parts be joined to the apex, the partial triangles thus formed are equivalent to each other.*

Corollary IV. *The diagonals of a parallelogram divide it into four equivalent triangles,* and conversely.

Corollary V. *Of the four triangles formed by the sides and diagonals of a trapezium, those formed by the lateral sides are equivalent to each other,* and conversely.

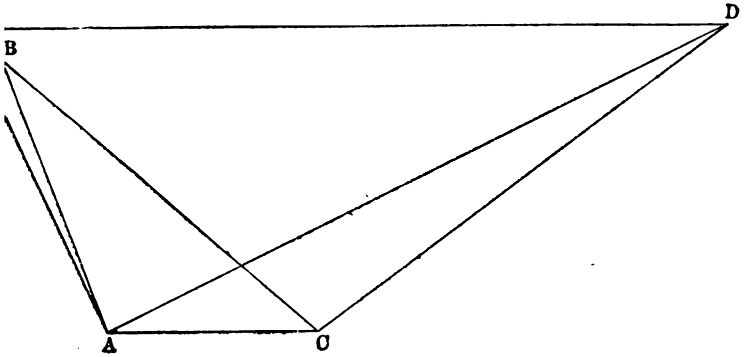
Scholium. The perimeters of two equivalent triangles having equal bases may differ infinitely.

Theorem 5.

If two equivalent triangles have equal bases, they have equal altitudes.

Let there be two equivalent triangles, A B C and A D C, having equal bases.

Let these triangles be applied to each other so that the bases coincide with each other in A C; through the apex D, let there be a parallel to A C, cutting C B or its prolongation in some point E; and let A E be joined. Because the triangles A E C and A D C are between the parallels A C and E D, they are equivalent to each other [3 i]; because A B C is equivalent to A D C [*Hyp.*], the triangles A B C and A E C are equivalent to each other [i]; therefore the triangle A B E has no magnitude: that is, B and E are one and the same point. Be-ause the vertices B and D are on the same parallel to A C, they are



qually distant from A C [I. 19]; therefore the two triangles A B C nd A D C have equal altitudes, W. W. T. B. D. Inversely

If two equivalent triangles have unequal altitudes, they have unequal ases.

Corollary I. *If two equivalent triangles have the same base or qual bases, on a straight line and on the same side thereof, they are etween the same parallels.* (Eucl. I. 39 and 40.)

Corollary II. *If two equivalent triangles have equal altitudes, they ave equal bases.*

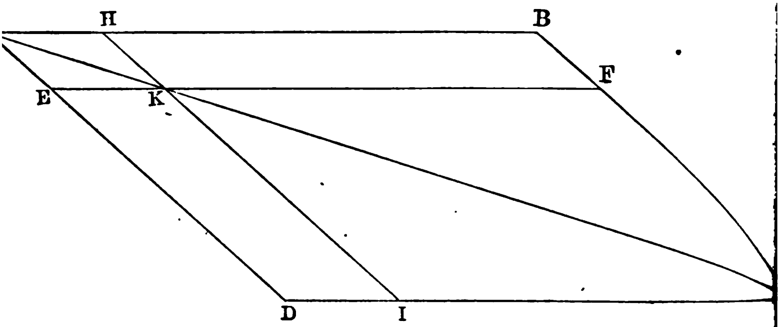
Theorem 6.

(*Eucl.* I. 43.)

The complements of two diagonal parallelograms are equi-alent to each other.

Let A E K H and K I C F be two diagonal parallelograms, and let I B F K and E D I K be their complements.

Let A K C be the diagonal of the parallelogram D B; then A K is he diagonal of the parallelogram E H, and K C that of the parallel-gram I F [III. *Def.* 21]. Because the triangle A H K is equivalent to he triangle A E K [III. 29 i], and the triangle K F C is equivalent to he triangle K I C, the triangles A H K and K F C are together equivalent to the triangles A E K and K I C [ii]: but the triangle A B C is equivalent to the triangle A D C; therefore the difference between



the triangle A B C and the sum of the triangles A H K and K F C is equivalent to the difference between the triangle A D C and the sum of the triangles A E K and K I C [iv]; that is, the parallelograms K B and K D are equivalent to each other, W. W. T. B. D.

Corollary. *If two straight lines be drawn parallel to the sides of a parallelogram through any point of a diagonal, the partial parallelograms determined by one of them, are equivalent to those determined by the other.*

SECTION II.

ON EQUIVALENT RIGHT-ANGLED QUADRANGLES.

Definitions.

3. The *square* of a straight line is the square having that line for one of its sides.

It has been shown [III. 39 i] that the squares of equal straight lines are not only equivalent, but equal to each other.

4. The *rectangle* of two straight lines, or of one straight

line by another, is the rectangle having these lines for its adjacent sides.

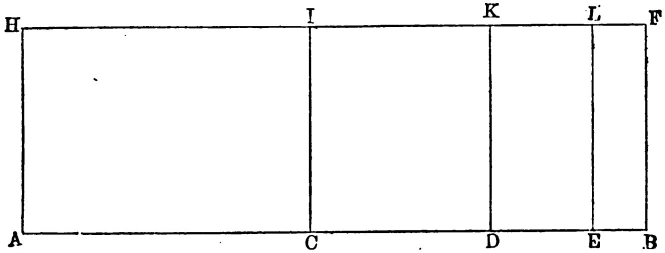
The sides adjacent to the base of a rectangle are perpendicular to that base [III. *Def.* 19]; hence, two rectangles having equal bases and altitudes are not only equivalent, but equal to each other [III. 39 ii]. Also, if two straight lines be equal to two others, each to each, the rectangle of the first two is equal to that of the others.

THEOREM 7.

(*Eucl.* II. 1.)

The rectangle of a straight line by the sum of several straight lines, is equivalent to the sum of the rectangles of the first line by each of the others.

Let a straight line A B be the sum of the straight lines A C, C D, D E and E B; let there be another straight line B F, and let A B F H be the rectangle of A B and B F.



Let C I, D K and E L be perpendiculars to A B; then A I, C K D L and E F are rectangles [III. *Def.* 19], and E F is the rectangle of E B by B F [*Def.* 4.]. Because C I, D K, E L are each equal to B F [I. 19], the rectangles A I, C K, D L are respectively equal to the rectangles of A C, C D, D E by B F; therefore the rectangle A F is equivalent to the sum of the rectangles of A C, C D, D E and E B by B F, which W. T. B. D.

Corollary I. *The square of the sum of several straight lines, is equivalent to the sum of its rectangles by each line.* (Eucl. II. 2.)

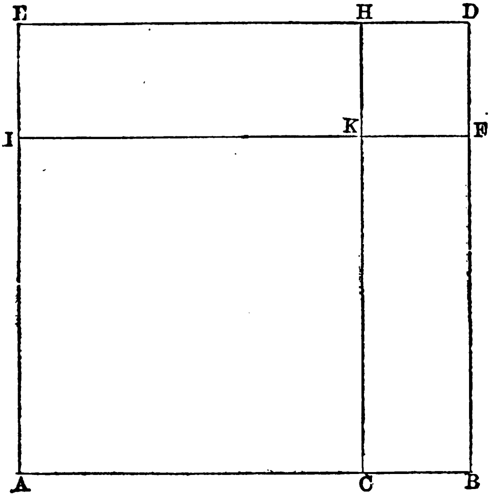
Corollary II. *The rectangle of the sum of several straight lines by one of them, is equivalent to the square of the latter together with its rectangle by the sum of the others.* (Eucl. II. 3.)

Corollary III. *The rectangle of the sum of several straight lines by the sum of several other straight lines, is equivalent to the sum of the rectangles of each line of the first sum by each line of the second.*

Theorem 8.

(*Eucl.* II. 4.)

The square of the sum of two straight lines, is equivalent to the sum of their squares together with twice their rectangle.



Let a straight line A B be the sum of the straight lines A C and C B, and let A B D E be the square of A B.

Let C H be a perpendicular to A B; let A I be equal to A C and let I F be a perpendicular to A E, cutting C H in a point K: then A K is the square of A C [*Def.* 3], and C F is equivalent to the rectangle of A C by C B. Because I E and C B are equal to each other [iv], H F is equivalent to the square of C B, and I H is equivalent to the rectangle of A C by C B; therefore the square A D is equivalent to the

of A C and C B, together with twice the rectangle of these
V. W. T. B. D.

LLARY I. *The square of a straight line is equivalent to four*
e square of its half, or the fourth part of the square of its

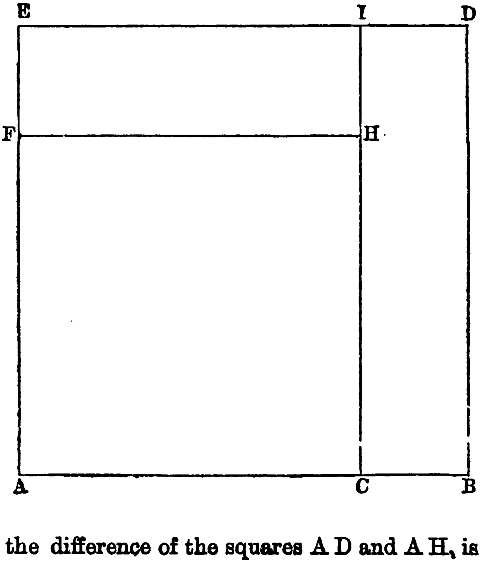
LLARY II. *The square of the sum of three straight lines,*
ent to the sum of their squares, together with twice the rectangles
f them by the others, and twice the rectangle of the latter.

THEOREM 9.

(*Eucl.* II. 5.)

difference of the squares of two straight lines, is equiva-
the rectangle of their sum by their difference.

B and A C be two straight lines; let A B D E be the square
and A C H F that of A C.



C H be pro-
to a point
E D; then
fference of
uares A D
H is com-
of the rect-
D and F I.
DB is
o A B, the
le C D is
ent to the
le of A B
; because
qual to C B
e rectangle
equivalent
ectangle of
C B: therefore the difference of the squares A D and A H, is

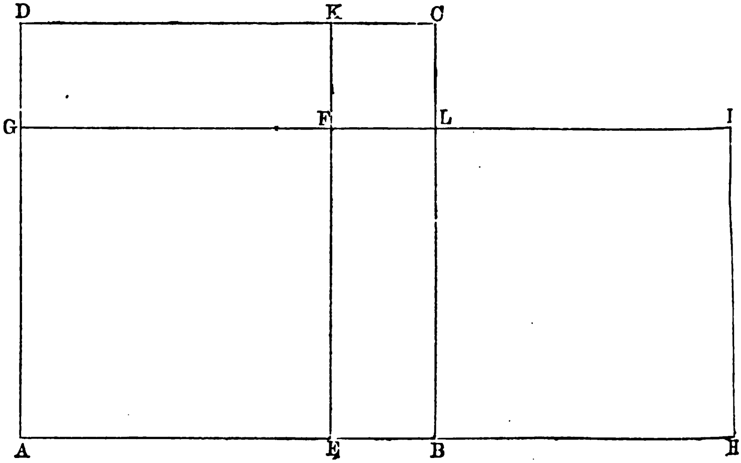
equivalent to the sum of the rectangles of A B and A C by C B. But the sum of the rectangles of A B and A C by C B is equivalent to the rectangle of the sum of A B and A C by C B [7], and C B is the difference of the lines A B and A C; therefore the difference of the squares of A B and A C is equivalent to the rectangle of their sum by their difference, W. W. T. B. D.

COROLLARY. *The rectangle of the difference of two straight lines by one of them, is equivalent to the difference between the square of the latter and the rectangle of the two lines.*

THEOREM 10.

The sum of the squares of two straight lines, is equivalent to the square of their difference together with twice their rectangle.

Let A B C D and A E F G be the squares of the straight lines A B and A E.



Let A B be produced to a distance B H equal to A E, and let H I be a perpendicular to A H, also equal to A E; let E F be produced to a point K of D C and let F I be joined, cutting B C in L: then

B H I L is equivalent to the square A F, and the figure A H I L C D is equivalent to the sum of the squares of A B and A E. Because A D is equal to A B, and A G is equal to A E, the line G D is equal to E B [iv]: but K C is equal to E B [I. 19], and C L is equal to D G; therefore K F L C is equivalent to the square of E B. Because B H is equal to A E [*Const.*], the line E H is equal to A B [ii]; but A D is equal to A B: therefore the rectangles E I and A K are equivalent to each other. Now, the rectangles A K and E I, or twice the rectangle A K, together with the square K L, are equivalent to the sum of the squares A C and B I; also, A K is equivalent to the rectangle of the line A B by A E, and K L is equivalent to the square of their difference; therefore the sum of the squares of A B and A E is equivalent to the square of their difference together with twice their rectangles, W. W. T. B. D.

COROLLARY. *The square of the difference of two straight lines, is equivalent to twice their rectangle less than the sum of their squares.*

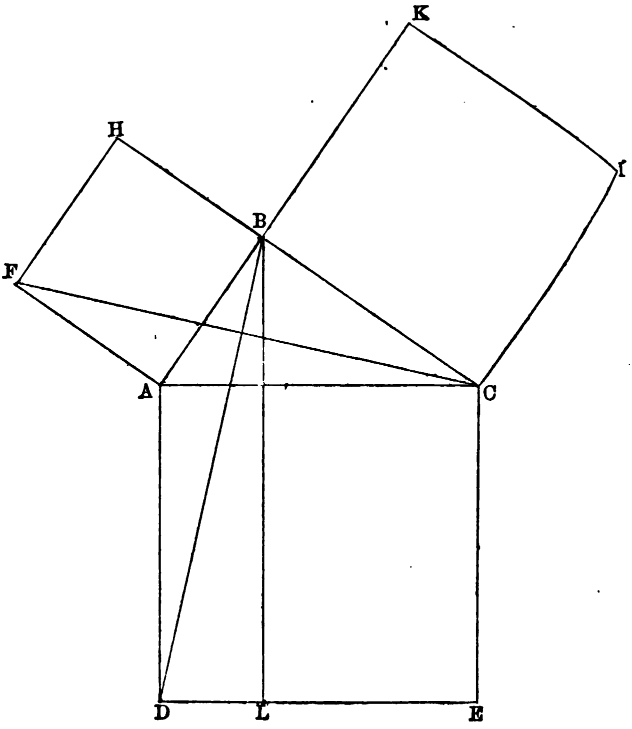
THEOREM 11.

(*Eucl.* I. 47.)

In a right-angled triangle, the square of the hypothenuse is equivalent to the sum of the squares of the other sides.

Let A C be the hypothenuse of a right-angled triangle A B C, and let A D E C be its square; let A F H B be the square of A B, and B K I C that of B C.

Let B D and F C be joined, and let B L be a perpendicular on D E; then B L is parallel to A D. Because the angles F A C and B A D have a part in common, and the parts which are not common are right-angles, they are equal to each other [viii]; because A F and A B are two sides of the square A H, they are equal to each other [III. *Def.* 18]; also A C is equal to A D: therefore the triangles F A C and B A D are equivalent to each other [III. 17]. Because the triangle F A C and the square A H have the same base A F and are between the parallels C H and A F, the triangle F A C is equivalent to half the square A H [3]; because the rectangle A L and the triangle B A D have the same base A D and are between the parallels A D and B L, the triangle B A D is equivalent to half the rectangle



A L: but the triangles F A C and B AD are equivalent to each other [*Dem.*]; then half the square A H is equivalent to half the rectangle A L; therefore the square A H is equivalent to the rectangle A L [vi]. The same construction and reasoning would prove that the square B I is equivalent to the rectangle C L; thence the rectangles A L and C L are together equivalent to the squares A H and B I [iv]: but the rectangles A L and C L together are the square A E; therefore the square A E is equivalent to the sum of the squares A H and B I, which W. T. B. D.

LLARY I. *The square of each right side of a right-angled*
, is equivalent to the rectangle of the hypothenuse by its pro-
on the latter.

LLARY II. *The square of each right side of a right-angled*
, is equivalent to the difference between the square of the hypo-
and that of the other side.

LLARY III. *The square of the altitude upon the hypothenuse of a*
gled triangle, is equivalent to the rectangle of the projections of
t sides on the hypothenuse.

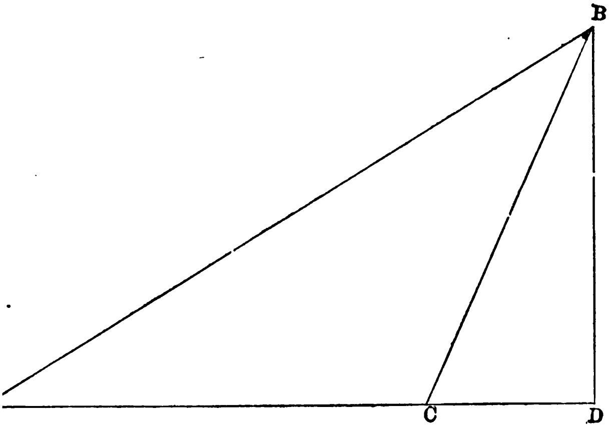
LLARY IV. *The rectangle of the right sides of a right-angled*
, is equivalent to the rectangle of the hypothenuse by the
of the opposite vertex.

LLARY V. *A square is half the square of its diagonal.*

THEOREM 12.

(*Eucl.* II. 12.)

square of the side subtending an obtuse angle of a
le, is equivalent to the squares of the other sides together
wice the rectangle of one of the latter by the projection of
ler on the prolongation of the first.



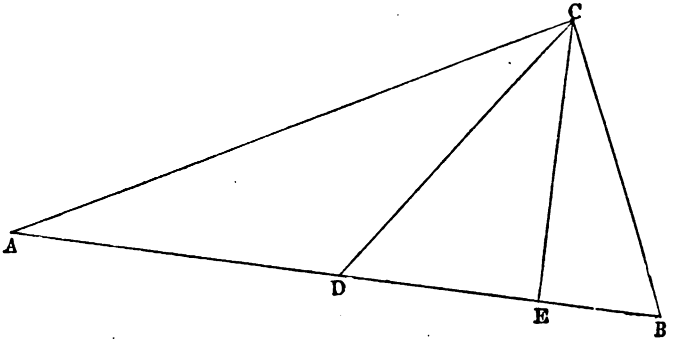
A B subtend the obtuse angle C in a triangle A B C.

Let A C be produced and let B D be a perpendicular on the prolongation of A C; then C D is the projection of B C on A D [I. *Def.* 19]. Because A B is the hypothenuse of the right-angled triangle A B D, the square of A B is equivalent to the squares of B D and A D [11]: but the square of A D is equivalent to the squares of A C and C D, together with twice the rectangle of A C by C D [8]; therefore the square of A B is equivalent to the squares of B D, C D and A C together with twice the rectangle of A C by C D. Now the squares of B D and C D are together equivalent to that of B C [11]; therefore the square of A B is equivalent to the squares of B C and A C together with twice the rectangle of A C, by the projection of B C on the prolongation of A C, which W. T. B. D.

COROLLARY. *The square of the side subtending an acute angle of a triangle, is equivalent to the difference between the squares of the other sides, and twice the rectangle of one of the latter by the projection thereon of the other.* (Eucl. II. 13.)

THEOREM 13.

The squares of two sides of a triangle, are together equivalent to twice the square of half the other side, with twice that of the median thereto.



Let A B C be a triangle, and C D the median to the side A B.

Let C E be a perpendicular to A B; then D E is the projection of C D on A B. Because the square of A C is equivalent to those of A D and C D, together with twice the rectangle of A D by D E [12]; and because the square of C B with twice the rectangle of D B by D E, is equivalent to the squares of D B and C D [12 i]; the squares of A C and C B with twice the rectangle of A D by D E, are together equivalent to the squares of A D and D B with twice that of C D, and twice the rectangle of D B by D E: but, because A D is equal to D B [*Const.*], the square of A D is equivalent to that of D B [III. 39 i], and the rectangle of A D by D E is equivalent to the rectangle of D B by D E [III. 39 ii]; therefore the squares of A C and C B are together equivalent to twice those of A D and C D [iv], which W. T. B. D.

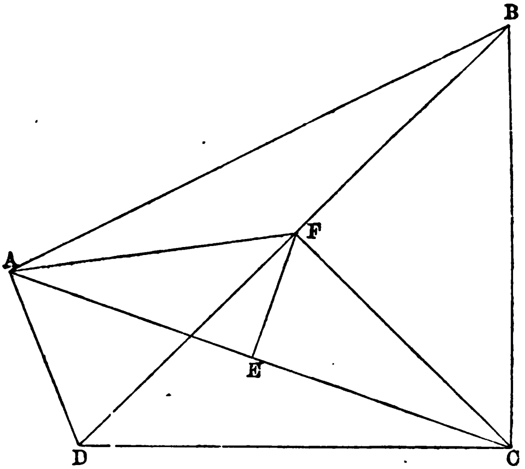
Scholium. The squares of the equal sides of an isosceles triangle, are together equivalent to twice the square of half the base, with twice that of the altitude.

THEOREM 14.

The squares of the sides of a quadrangle, are together equivalent to those of its diagonals, with four times that of the middle line of the diagonals.

Let there be a quadrangle A B C D and let E F be the middle line of its diagonals A C and B D.

Let A F and F C be joined; then F E is a median of the triangle A F C. Because the squares of A B and A D are together equivalent to twice those of D F and A F [12], and because the squares of B C and C D are together equivalent to twice those of D F and F C, the squares of A B, A D, B C, and C D are together equivalent to four times the square of D F with twice those of A F and F C [vi]. Because the squares of A F and F C are together equivalent to twice those of A E and E F, twice the squares of A F and F C are equivalent to four times those of A E and E F [vi]; therefore the squares of A B, A D, B C and C D are together equivalent to four times those of D F, A E and E F: but four times the squares of D F and A E are equivalent to the squares of D B and A C [8 i];



therefore the squares of the sides of the quadrangle A B C D, are together equivalent to those of the diagonals D B and A C, with four times that of their middle line E F, which W. T. B. D.

COROLLARY. *The squares of the sides of a parallelogram, are together equivalent to those of its diagonals.*

SECTION III.

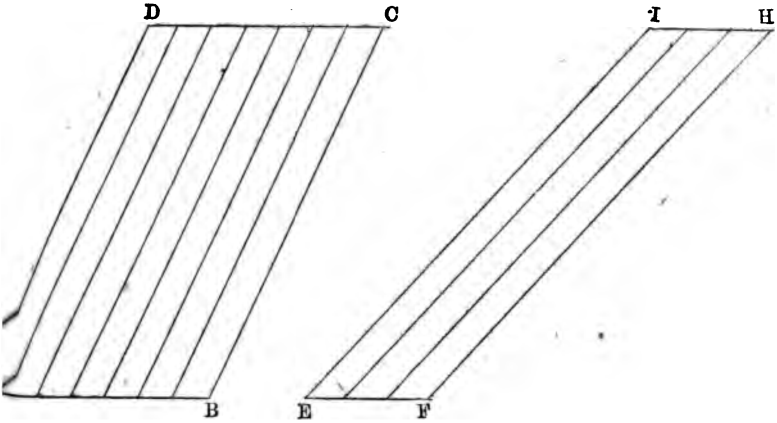
ON THE COMPARISON OF THE SIDES OF EQUIVALENT PLANE FIGURES.

THEOREM 15.

(*Eucl.* VI. 1.)

Two parallelograms having equal altitudes are as their bases.[1]

Let A B C D and E F H I be two parallelograms having equal altitudes.



Let the bases A B and E F be divided into equal parts small enough to be contained an equal number of times in each one; let, for instance, A B contain *seven*, and E F *three* of these parts: then A B is to E F as *seven* is to *three* [x].

Now, let there be drawn, through each point of division of A B and E F, a parallel to the adjacent sides. Then the two parallelograms A C, E H, are divided into parallelograms having all equal bases and altitudes, and being consequently all equivalent to one another [1]; but

---

[1] In The Elements, the expression that two or more figures *are* as two or more others, is used with the meaning that the first *are proportional* to the others.

A C will contain *seven*, and E H *three* of these parallelograms; then the parallelogram A C is to the parallelogram E H as *seven* is to *three* [x].

Therefore, the parallelogram A C is to the parallelogram E H as the base A B is to the base E F [i].

Should the lines A B and E F be incommensurable, let them be imagined to be divided into their infinitely small elements, and let a parallel to the adjacent sides be imagined through each point of division; then the two parallelograms will be divided into a number, finite or infinite, of parallelograms having equal bases [*Post.* IV] and altitudes [*Hyp.*]; therefore these partial parallelograms will be equivalent to one another [1]. However small these partial parallelograms may be, it is evident that the number of them in each of the parallelograms A D and E H, will be the same as that of the parts into which A B and E F have been divided; therefore the two parallelograms are still to each other as their bases W. W. T. B. D.

COROLLARY I. *Two triangles having equal altitudes are as their bases.*

COROLLARY II. *Two parallelograms, or triangles, which are between the same parallels, are as their sides thereon.*

COROLLARY III. *Two parallelograms, or triangles, having equal bases, are as their altitudes.*

COROLLARY IV. *The rectangles of a straight line by two others are as the latter.*

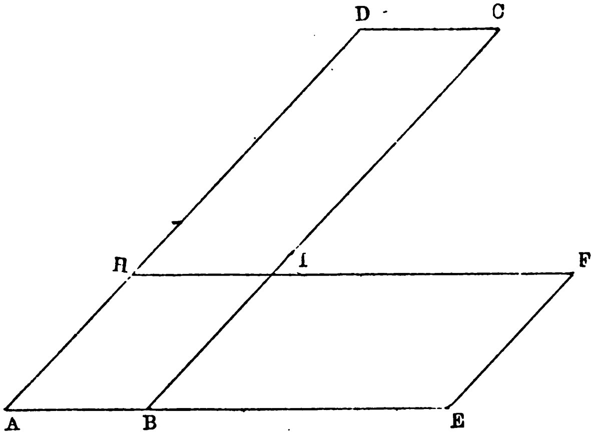
COROLLARY V. *Two parallelograms having equal altitudes, are as the rectangles of their bases by the same straight line.*

THEOREM 16.

(*Eucl.* VI. 14.)

The sides of the homologous angles of two equivalent equiangular parallelograms, are reciprocally proportional.

Let there be two equiangular parallelograms A B C D and A E F H equivalent to each other, and let D A B and H A E be two homologous angles.



Let the two parallelograms be applied to each other so that the sides A B and A E, and the homologous angles D A B and H A E, coincide with each other respectively; and let I be the point where F H cuts B C: then the parallelograms A C and A F have the part A B I H in common. Because the parallelograms A F and A I are between the parallels A E and H F, they are as their sides on A E [15 ii]; that is, A F is to A I as A E is to A B; also, because the parallelograms A C and A I are between the parallels A D and B C, the parallelogram A C is to A I as A D is to A H: but the parallelograms A F and A C are equivalent to each other [*Hyp.*]; therefore A E is to A B as A D is to A H [i], which W. T. B. D. Inversely,

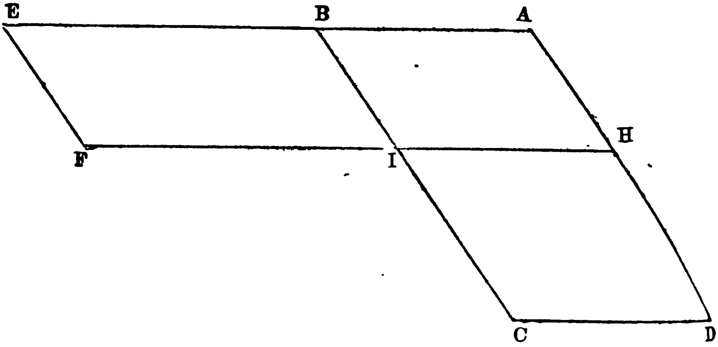
If the sides of homologous angles of two equiangular parallelograms be not reciprocally proportional, the two parallelograms are not equivalent to each other.

THEOREM 17.

(*Eucl.* VI. 14.)

Conversely, *if two equiangular parallelograms have the sides of homologous angles reciprocally proportional, they are equivalent to each other.*

Let there be two equiangular parallelograms A B C D and A E F H; let B A D and E A H be two homologous angles, and let A E be to A B as A D is to A H.



Let these parallelograms be applied to each other so that the sides A E and A B, and the homologous angles B A D and E A H, coincide respectively with each other; and let I be the point where F H cuts B C: then the parallelograms A C and A F have the part A B I H in common. Because the parallelograms A F and A I are between the parallels A E and H F, they are as their sides on A E [15 i]; that is, A F is to A I as A E is to A B; also, because the parallelograms A C and A I are between the parallels A D and B C, the parallelogram A C is to A I as A D is to A H: but A E is to A B as A D is to A H [*Hyp.*]; then A F is to A I as A C is to A I [i]; therefore the parallelograms A F and A C are equivalent to each other [xvi], W. W. T. B. D. Inversely,

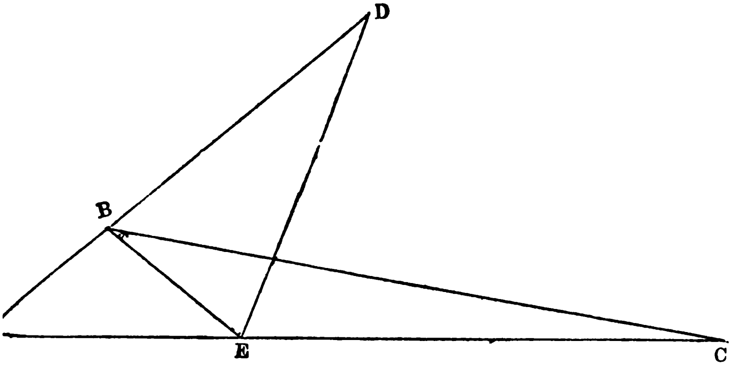
If two equiangular parallelograms be not equivalent to each other, the sides of homologous angles are not reciprocally proportional.

THEOREM 18.

(*Eucl.* VI. 15.)

Two equivalent triangles which have an angle of the one equal to an angle of the other, have the sides of these angles reciprocally proportional.

Let there be two equivalent triangles, A B C and A D E, having the angle B A C of the one equal to the angle D A E of the other.



Let the two triangles be applied to each other so that the equal angles coincide with each other, and let B E be joined; then the triangles A B C and A D E have the part A B E in common. Because the triangles A B C and A B E have the same altitude, they are as their bases [15 i]; that is, A B C is to A B E as A C is to A E; also, because the triangles A D E and A B E have the same altitude, A D E is to A B E as A D is to A B: but the triangles A B C and A D E are equivalent to each other [*Hyp.*]; therefore A C is to A E as A D is to A B [i], which W. T. B. D.

COROLLARY I. *Two equivalent triangles which have an angle of the one supplementary to an angle of the other, have the sides of these angles reciprocally proportional.*

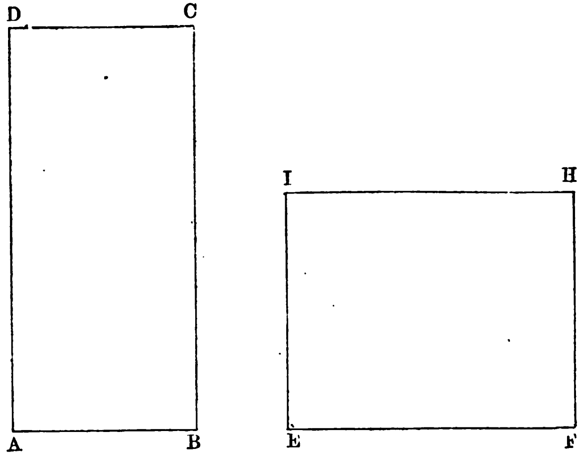
Corollary II. *If two triangles have an angle of the one equal, or supplementary, to an angle of the other, and if they have the sides of these angles reciprocally proportional, they are equivalent to each other.* (Eucl. VI. 15.)

Theorem 19.

(*Eucl.* VI. 16.)

The bases and altitudes of two equivalent rectangles are reciprocally proportional.

Let A B C D and E F H I be two equivalent rectangles.



Because the two rectangles are equiangular to each other, their adjacent sides are reciprocally proportional [16]: but two adjacent sides of each rectangle are its base and altitude; therefore the bases and altitudes of the two rectangles are reciprocally proportional, W. W. T. B. D.

Corollary I. *Each side of a square equivalent to a rectangle, is a mean proportional to the adjacent sides of the latter.* (Eucl. VI. 17.)

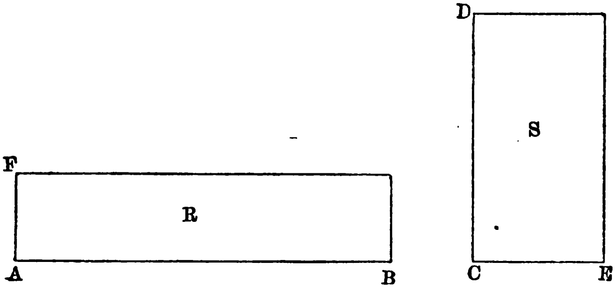
Corollary II. *The bases and altitudes of two equivalent parallelograms, or triangles, are reciprocally proportional.*

Theorem 20.

(*Eucl.* VI. 16.)

Conversely, *if four straight lines be proportional, the tangle of the means is equivalent to that of the extremes.*

Let there be four straight lines A B, C D, C E and A F, and let B be to C D as C E is to A F.



Let R be the rectangle of the lines A B and A F, and let S be the ctangle of C D and C E; then these rectangles have their adjacent les reciprocally proportional [*Hyp.*]. Because the angles of the ctangles R and S are all right angles, the two rectangles are equigular to each other; therefore they are equivalent to each other 7], W. W. T. B. D.

Corollary I. *If three straight lines be proportional, the square of the ean is equivalent to the rectangle of the two other lines.* (Eucl. VI. 17.)

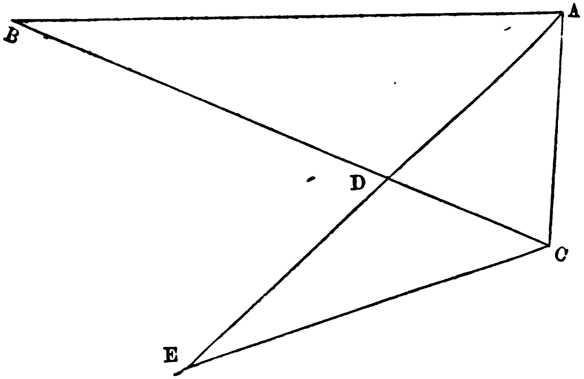
Corollary II. *Two parallelograms, or triangles, which have their ses and altitudes reciprocally proportional, are equivalent to each other.*

Theorem 21.

The rectangle of two sides of a triangle is equivalent to the uare of the bisectrix of their angle, together with the rectangle the portions of the third side, determined by the bisectrix.

Let A B C be a triangle, and A D the bisectrix of the angle B A C, tting B C in a point D.

Let A D be produced to some point E, and let the angle D C E be made equal to the angle B A D: then the angle D E C is equal to the angle A B D [III. 3 ii]. Because the triangles A B D and A E C have the angles B A D and A B D, respectively equal to the angles C A E and A E C, they are similar to each other [III. 23]; thence, A B is to A E as A D is to A C [III. 22], and the rectangle of A B and A C is equivalent to that of A E and A D [20]: but the rectangle of A E and A D is equivalent to the square of A D, together with the rectangle of A D and D E [7 ii]; therefore, the rectangle of A B and A C is equivalent to the square of A D, together with the rectangle of A D and D E [i]. Because the triangles A B D and D C E are also similar to each other [III. 23], B D is to D E as A D is to D C; hence the rectangle of B D and D C is equivalent to that of A D and D E [20]: therefore, the rectangle of A B and A C is equivalent to the square of A D, together with the rectangle of B D and D C, which W. T. B. D.

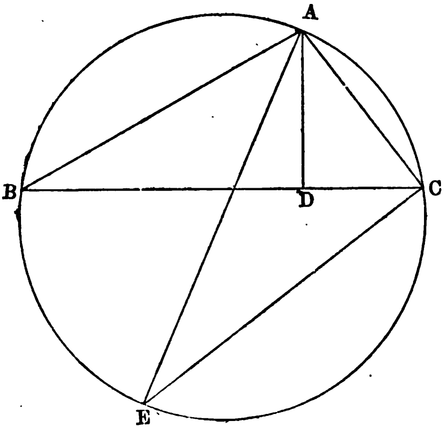


Corollary. *The rectangle of two sides of a triangle, together with the square of the bisectrix of their exterior angle, are equivalent to the rectangle of the distances from the extremities of the third side, to its intersection with the bisectrix.*

THEOREM 22.

The rectangle of the altitude upon one side of a triangle by the diameter of the circumscribed circle, is equivalent to that of the other sides.

Let A D be the altitude upon the side B C of the triangle A B C inscribed in the circle A B E.



Let A E be a diameter of the circle A B E and let E C be joined. Because the angle A C E is inscribed in a semicircle, it is a right angle [III. 77]; because the angles A B C and A E C are inscribed in the same segment, they are equal to each other [III. 78 iii], and the two right-angled triangles A B D and A E C are similar to each other: thence AB is to A E as A D is to A C; therefore the rectangle of A B by A C is equivalent to A E by A D [20], which W. T. B. D.

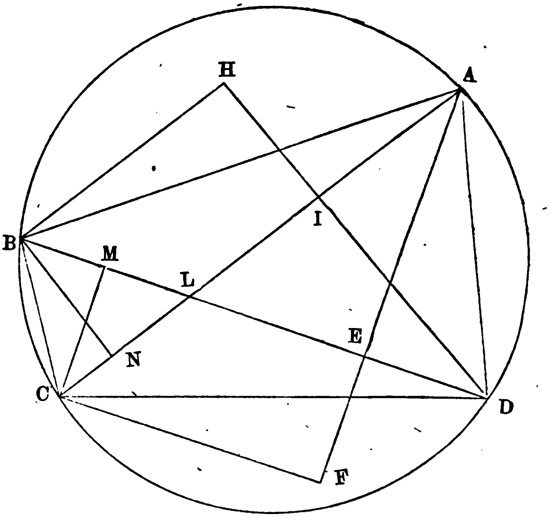
COROLLARY I. *If the diagonals of a quadrangle inscribed in a circle be perpendicular to each other, the rectangle of each diagonal by the diameter of the circle, is equivalent to the sum of the rectangles of the sides adjacent to its extremities.*

COROLLARY II. *The rectangle of the sides of the angle at the apex of a triangle, and that of the sides of the angle at the apex of another triangle inscribed in the same, or an equal circle, are to each other as the altitudes.*

THEOREM 23.

The diagonals of a quadrangle inscribed in a circle, are as the sums of the rectangles of the sides adjacent to their extremities.

Let A B C D be a quadrangle inscribed in the circle A C D, and let A C and B D be its diagonals.



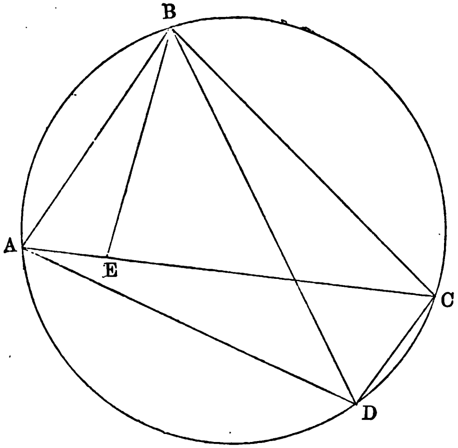
Let A E and C M be the altitudes of the triangles A B D and B C D upon their common side B D, and let D I and B N be the altitudes of the triangles A D C and A B C upon their common side A C. Because the rectangle of A B by A D is equivalent to that of A E by the diameter of the circle [22], and the rectangle of B C by C D is equivalent to that of C M by the diameter of the circle, the rectangles of A B by A D and B C by C D, are together equivalent to that of the sum of A E and C M, by the diameter of the circle [iv]; likewise, the rectangles of A D by D C and A B by B C, are together equivalent to that of the sum of D I and B N, by the diameter of the circle:

therefore, the sum of the rectangles of A B by A D and B C by C D, is to that of the rectangles of A D by D C and A B by B C, as the sum of A E and C M is to that of D I and B N [15 iv].

Let A E be produced to a distance E F equal to C M, and let D I be produced to a distance I H equal to B N; also let C F and B H be joined: then A F and D H are perpendicular to C F and B H respectively; also, the sum of the rectangles of A B by A D and B C by C D, is to that of the rectangles of A D by D C and A B by B C, as A F is to D H. Because the angles D B H and A C F are each equal to the angle B L C [II. 20], they are equal to each other [i]; hence, the right-angled triangles A C F and D B H are similar to each other [III. 23 ii], and A C is to D B as A F is to D H [III. 22]: therefore, the sum of the rectangles of A B by A D and B C by C D, is to that of the rectangles of A D by D C and A B by B C, as A C is to D B [i], which W. T. B. D.

THEOREM 24.

The rectangle of the diagonals of a quadrangle inscribed in a circle, is equivalent to the sum of the rectangles of its opposite sides.



Let A B C D be a quadrangle inscribed in the circle A C D, and let A C and B D be its diagonals.

Let the angle A B E be equal to the angle D B C, and let B E cut A C in a point E; then the angles C B E and A B D are equal to each other [viii]. Because the angle A C B is equal to the angle A D B [III. 78 iii], the triangles A B D and E B C are similar to each other [III. 23]; hence B C is to B D as C E is to A D [III. 22]: therefore the rectangle of B C and A D is equivalent to that of B D and C E [20]. Because the angle B A C is equal to the angle B D C [III. 78 iii], the triangles A B E and B C D are similar to each other [III. 23]; hence A B is to B D as A E is to C D; therefore the rectangle of A B and C D is equivalent to that of B D and A E: but the rectangle of B C and A D is equivalent to that of B D and C E [*Dem.*]; then the rectangle of A B by C D and of B C by A D, are together equivalent to the sum of the rectangles of B D by A E and B D by C E [iv]; therefore, they are equivalent to the rectangle of B D by A C [7], which W. T. B. D.

Corollary. *If two adjacent sides of a quadrangle inscribed in a circle be equal to each other, the rectangle of the diagonals is equivalent to that of one of the equal sides by the sum of the others.*

SECTION IV.

ON PROPORTIONAL AREAS.

Definition.

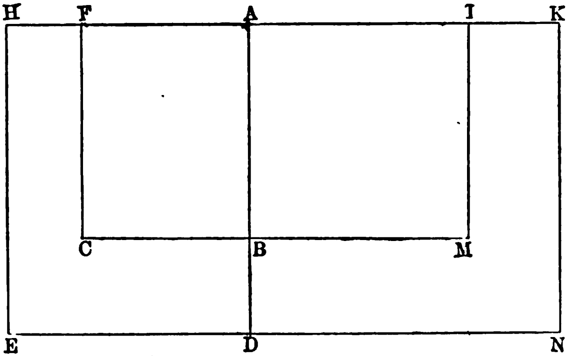
5. A *plane lune,* or simply *lune,* is the plane figure limited by two arcs having the same extremities.

A lune is designated by the letters at the extremities of its arcs, and one or more letters at some points of each arc.

Theorem 25.

If four straight lines be proportional, the squares of the homologous lines are as their rectangles by the corresponding ones.

Let a straight line A B be to a second B C as a third A D is to a fourth D E; let A C and A E be the rectangles of A B by B C and A D by D E respectively; also let A M be the square of A B and A N that of A D.



Let the rectangle A E and the square A N be placed on opposite sides of A D so that A D be a side common to both; also let the rectangle A C and the square A M be placed on opposite sides of A D, so that A B be a side common to both, and so that the four quadrangles have the same vertex A; then A I, A F, A K and A H are equal to A B, B C, A D and D E respectively. Because the square A M and the rectangle A C have the same altitude, A M is to A C as A I is to A F [15]; also A N is to A E as A K is to A H: but A I is to A F as A K is to A H [*Hyp.*]; then A M is to A C as A N is to A E [i]; therefore A M is to A N as A C is to A E [xiii], which W. T. B. D.

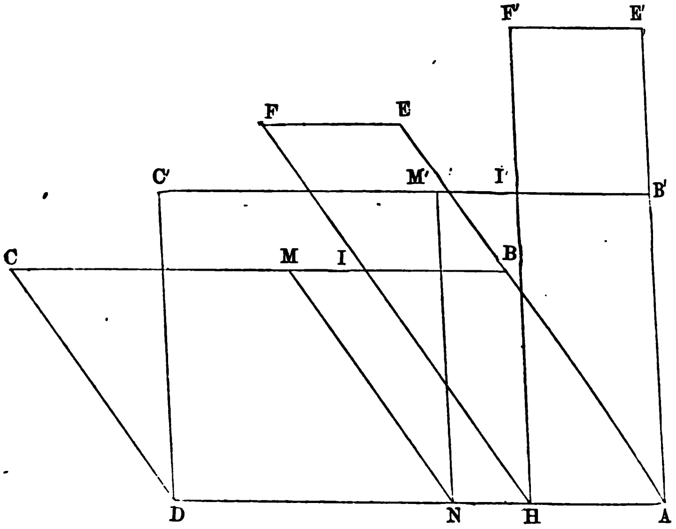
COROLLARY. *If four straight lines be proportional, the rectangles of the homologous lines are as the squares of the two first lines, or the two last ones.*

THEOREM 26.

(*Eucl.* VI. 23.)

Two equiangular parallelograms are as the rectangles of their adjacent sides.

Let A B C D and A E F H be two equiangular parallelograms and let A B′ C′ D and A E′ F′ H be the rectangles of their adjacent sides A B, A D and A E, A H, respectively.



Let the two parallelograms and the two rectangles be applied to each other so that A H coincides with A D, A B with A E, and A B′ with AE′; and let I and I′ be the points where H F and H F′ cut respectively C B and C′ B′. Let I M be to I B as I F is to I H; and let M N be a parallel to I H, and N M′ a parallel to H I′: then the parallelogram N I and I E are equivalent to each other [17]; also the rectangles N I′ and I′ E′ are equivalent to each other. Because the parallelograms A F and A M have a part A I in common, and the parts not common are equivalent, they are equivalent to each other [viii]; likewise the rectangle A F′ is equivalent to the rectangle A M′: now, the parallelograms A C and A M are as the rectangles A C′ and A M′ [15 v]; therefore the parallelograms A C and A F are as the rectangles A C′ and A F′, which W. T. B. D.

Corollary. *Two triangles having one angle of the one equal, or sup-*

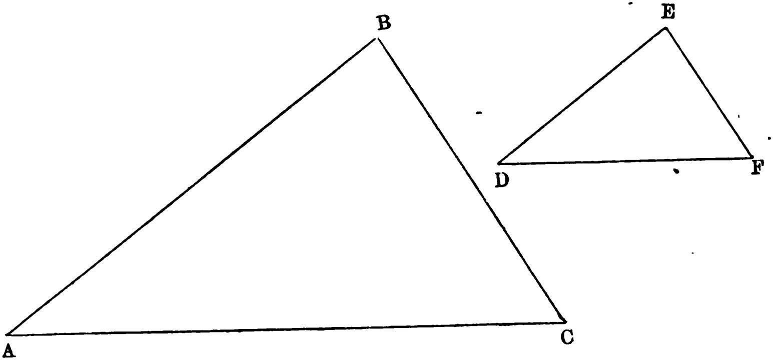
plementary, to one angle of the other, are as the rectangles of the sides of these angles.

THEOREM 27.

(*Eucl.* VI. 19.)

Two similar triangles are as the squares of their homologous sides.

Let there be two similar triangles, A B C and D E F, and let D, E and F be the vertices homologous to A, B and C respectively.



Because the angles A and D are equal to each other, the triangle A B C is to the triangle D E F as the rectangle of A B and A C is to that of D E and D F [26 i]: but A B is to D E as A C is to D F [III. 22]; then the rectangle of A B and A C is to that of D E and D F, as the square of A B is to that of D E [25 i]; therefore the triangle A B C is to the triangle D E F as the square of A B is to that of D E [i], which W. T. B. D.

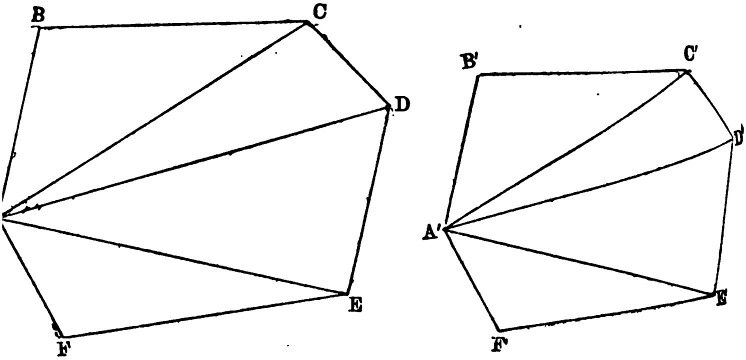
COROLLARY. *Two similar triangles are as the squares of the medians to homologous sides, or of the altitudes of homologous vertices.*

THEOREM 28.

(*Eucl.* VI. 20.)

The homologous triangles of two similar polygons, are proportional.

Let there be two similar polygons A B C D E F and A′ B′ C′ D′ E′ F′, and let triangles A B C, A C D, A D E and A E F be homologous to the triangles A′ B′ C′, A′ C′ D′, A′ D′ E′ and A′ E′ F′ respectively.



Because the triangles A B C and A′ B′ C′ are similar to each other [III. 41 v], A B C is to A′ B′ C′ as the square of A C is to that of A′ C′ [27]; likewise A C D is to A′ C′ D′ as the square of A C is to that of A′ C′; therefore A B C is to A′ B′ C′ as A C D is to A′ C′ D′ [i]. The same reasoning would prove that A D E is to A′ D′ E′ as A C D is to A′ C′ D′, and that A E F is to A′ E′ F′ as A D E is to A′ D′ E′; therefore, A B C is to A′ B′ C′ as A C D is to A′ C′ D′, as A D E is to A′ D′ E′, and as A E F is to A′ E′ F′, which W. T. B. D.

COROLLARY. *Two similar polygons are as their homologous triangles* (xiv).

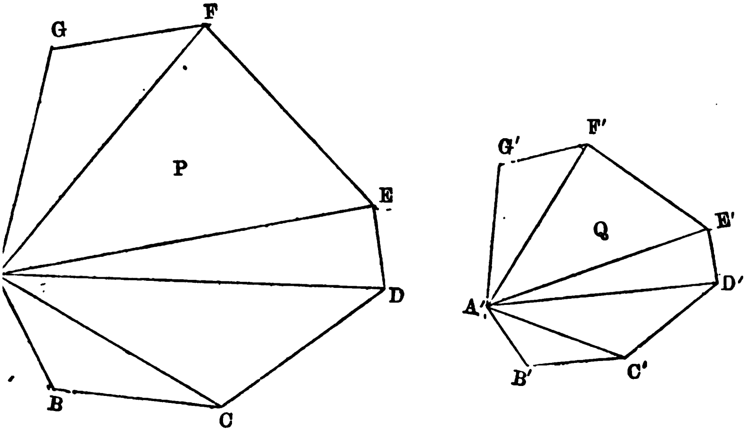
THEOREM 29.

(*Eucl.* VI. 20.)

Two similar polygons are as the squares of their homologous sides.

Let there be two similar polygons P and Q, and let A B C and A′ B′ C′ be two homologous triangles.

Because the polygons P and Q are similar to each other, they are as their homologous triangles [28 i]; thence P is to Q as A B C is to A′B′C′. Because the triangles A B C and A′ B′ C′ are similar to each other [III. 41 v], they are as the squares of their homologous sides [27]; hence A B C is to A′B′ C′ as the square of A B is to that of A′B′; therefore P is to Q as the square of A B is to that of A′ B′ [i], which W. T. B. D.



Corollary. *Two similar polygons are as the squares of their homologous diagonals, or of their perimeters.*

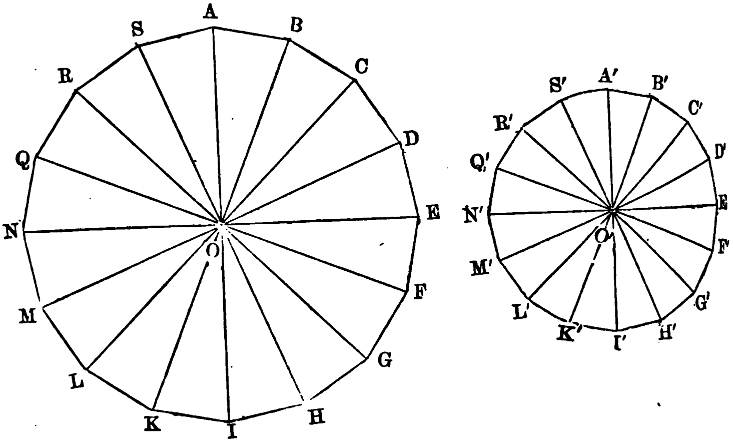
Theorem 30.

(*Eucl.* XII. 1.)

Two regular polygons of the same number of sides, are as the squares of their radii.

Let there be two regular polygons O and O′ having the same number of sides.

Let A O B be a sector triangle of the polygon O, and A′ O′ B′ a sector triangle of the polygon O′. Because the polygons O and O′ are similar to each other [III. 59 ii], the polygon O is to the polygon O′ as the triangle A O B is to the triangle A′ O′ B′ [28]: but the triangles A O B and A′ O′ B′ are similar to each other [III. 59 iii]; then A O B is to A′ O′ B′ as the square of O A is to that of O′ A′ [27]; therefore the polygon O is to the polygon O′ as the square of the radius O A is to the square of the radius O′ A′ [i], which W. T. B. D.



Corollary I. *Two regular polygons of the same number of sides, are as the squares of their apothems.*

Corollary II. *Two even-sided regular polygons of the same number of sides, are as the squares of their diameters.*

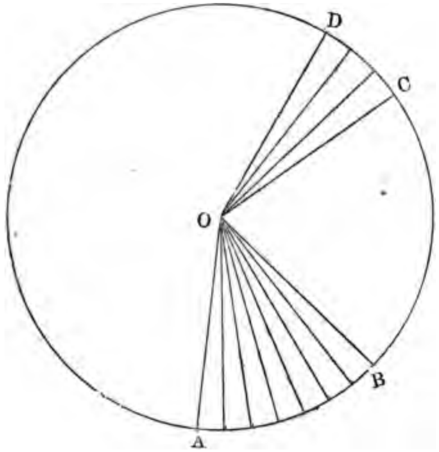
Corollary III. *Two circles are as the squares of their radii, or diameters.* (Eucl. XII. 2.)

Corollary IV. *Two similar polygons are as the squares of their homologous lines.*

Theorem 31.

Sectors of a circle are as their angles.

Let A O B and C O D be any two sectors of the circle O.



Let the arcs A B and C D be divided into equal parts, small enough to be contained an exact number of times in each: let, for instance, A B contain *seven,* and C D *three* of these parts; and let there be a radius to each point of division of both arcs. Because all the partial centre angles thus formed, stand upon equal arcs, they are equal to one another [II. 26]: but the angle A O B contains *seven,* and the angle C O D *three,* of these equal angles; therefore the angle A O B is to the angle C O D as *seven* is to *three* [x]. Because the partial sectors into which the two sectors have been divided, have equal angles, they are equal to one another [III. 76]: but the sector A O B contains *seven,* and the sector C O D *three,* of these partial sectors; then the sector A O B is to the sector C O D as *seven* is to *three* [x]; therefore the sector A O B is to the sector C O D as the angle A O B is to the angle C O D [i].

Should the arcs A B and C D be incommensurable, let them be imagined to be divided into their infinitely small elements, which are the same for both arcs [*Post.* IX], and to each point of division let a radius be imagined. The two angles A O B and C O D will be divided into a number, finite or infinite, of equal centre angles [II. 26]; therefore the partial sectors into which the two sectors have been divided are equal to one another [III. 76]. However small these partial sectors may be, it is evident that their number will be equal to that of the angles, in each of the sectors A O B and C O D; therefore the two sectors are as their angles, W. W. T. B. D.

Corollary I. *Sectors of equal circles are as their angles.*

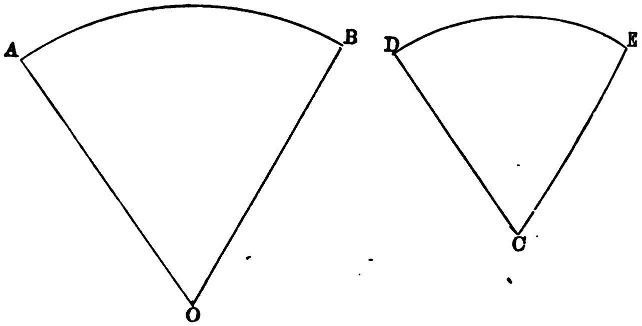
Corollary II. *Sectors of a circle, or of equal circles, are as their arcs.*

Scholium. A sector of a circle is to the whole circle as the arc is to the whole circumference, or as the angle is to four right angles.

Theorem 32.

Two similar sectors are as the squares of their sides.

Let A O B and D C E be two similar sectors.



Because the circle O is a sector of which the angle is equal to four right angles, the sector A O B is to the circle O as the angle A O B is to four right angles [31]; likewise the sector D C E is to the circle C as the angle D C E is to four right angles. Because the angles A O B and D C E are equal to each other [III. 81], the sector A O B is to the circle O as the sector D C E is to the circle C [i]; thence, the sector A O B is to the sector D C E as the circle O is to the circle C [xiii]: but the circle O is to the circle C as the square of O A is to that of C D [30 iii]; therefore the sector A O B is to the sector D C E as the square of O A is to that of C D [i], which W. T. B. D.

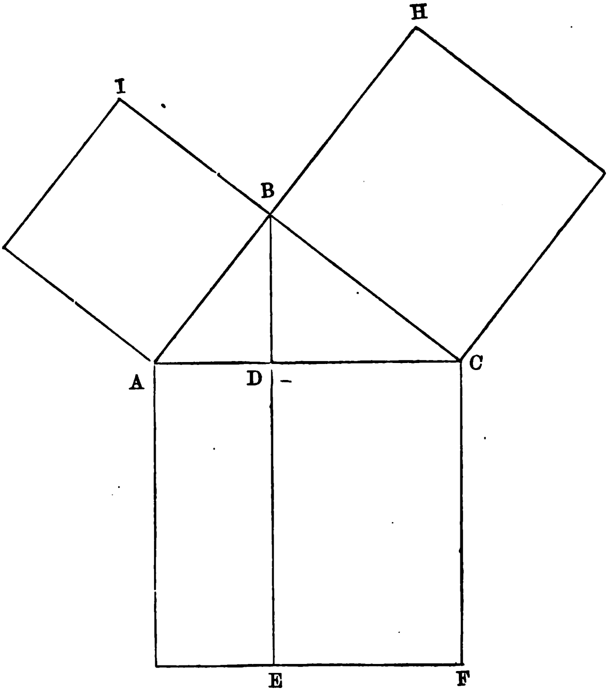
Corollary I. *Two similar sectors are to each other, as the squares of the chords of their arcs.*

COROLLARY II. *Two similar segments are to each other as the squares of their bases.*

Scholium. Two similar plane figures are to each other, as the squares of their homologous lines.

THEOREM 33.

The squares of the right sides of a right-angled triangle, are to each other as the projections of these sides on the hypothenuse.



Let A I and C H be the squares of the right sides A B and B C of the right-angled triangle A B C, and let B D be the altitude upon the hypothenuse.

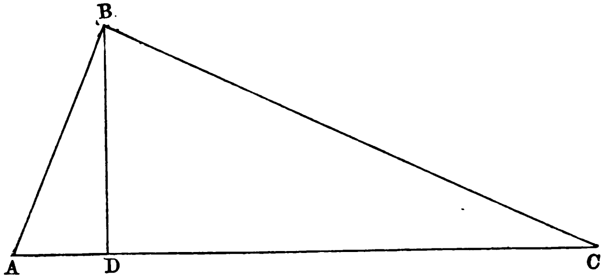
Let A E and D F be the rectangles of A C by A D and D C: then the bases A D and D C of these rectangles are the projections of the right sides A B and B C on the hypothenuse. Because the rectangles A E and D F have the same altitude, they are as their bases [15]; that is, A E is to D F as A D is to D C: but the squares A I and C H are respectively equivalent to the rectangles A E and D F [11 i]; therefore the square A I is to the square C H as A D is to D C [i], which W. T. B. D.

COROLLARY. *The square of each right side of a right-angled triangle, is to the square of the hypothenuse, as its projection on the hypothenuse is to the hypothenuse itself.*

THEOREM 34.

If the right sides of a right-angled triangle, be respectively equal to homologous lines of two similar plane figures, these plane figures are to each other, as the projections of the right sides on the hypothenuse.

Let there be two similar plane figures, P and Q, of which two homologous lines are respectively equal to A B and B C, and let A B C be a triangle right-angled in B.



Let B D be the altitude upon the hypothenuse A C; then A D and D C are the projections of A B and B C on the hypothenuse. Because the plane figures P and Q are similar to each other, P is to Q as the square of A B is to that of B C: but the square of A B is to that of B C as A D is to D C [33]; therefore P is to Q as A D is to D C [i], which W. T. B. D.

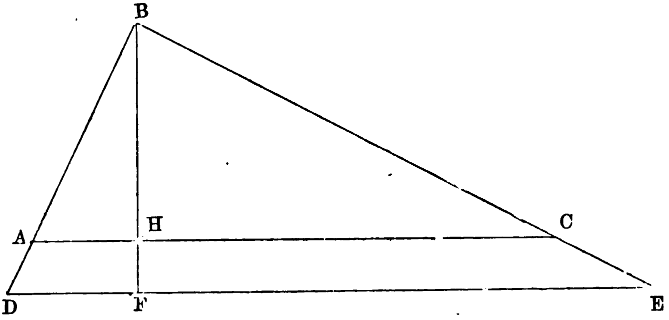
Corollary. *If the homologous lines of two similar plane figures, be respectively equal to a right side and the hypothenuse of a right-angled triangle, these plane figures are to each other as the projection of the right side on the hypothenuse is to the hypothenuse itself.*

Theorem 35.

(*Eucl.* VI. 22.)

If two similar plane figures have their homologous lines proportional to those of two other similar plane figures, they are proportional to the latter.

Let there be two similar plane figures P and Q, of which A B and B C are homologous lines; let there be two other similar plane figures R and S, of which D B and B E are homologous lines, and let A B be to B C as D B is to B E.



Let A B and B C form a right angle A B C, and from the vertex B let distances equal to B D and B E be taken on the sides B A and B C; let A C and D E be joined, and let D F be a perpendicular to D E cutting A C in H: then A C and D E are the hypothenuses of the right-angled triangles A B C and D B E. Because the lines A C and D E subtend the angle D B E proportionally [*Hyp.*], they are parallel to each other [II. 39]; therefore B H is perpendicular to A C [I. 17]. Because P and Q are similar to each other, P is to Q as A H is to H C [34]; also, because R and S are similar to each other, R is to S as D F is to F E: but A H is to H C as D F is to F E [II. 42]; therefore P is to Q as R is to S [i], which W. T. B. D.

COROLLARY. Conversely, *if two plane figures similar to each other, be proportional to two others similar to each other, the homologous lines of the two first, are proportional to those of the others.*

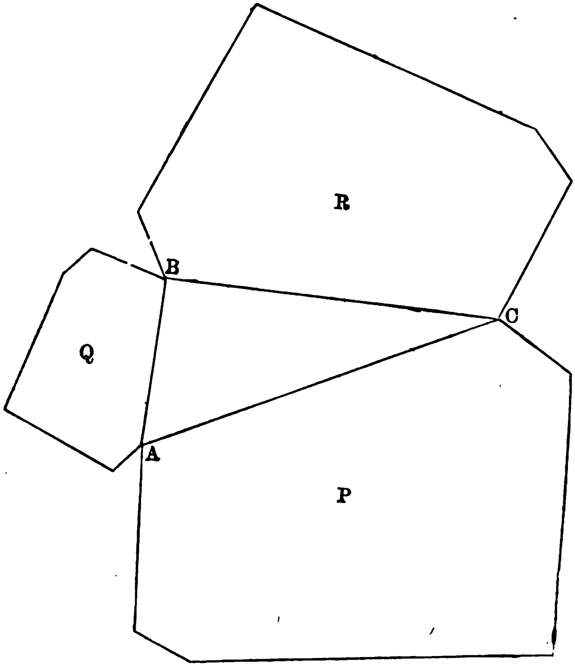
THEOREM 36.

(*Eucl.* VI. 31.)

If the sides of a right-angled triangle be the homologous sides of three similar polygons, that upon the hypothenuse is equivalent to the sum of the two others.

Let there be a right-angled triangle A B C, and three polygons, P, Q and R, similar to each other, of which the hypothenuse A C, and the right sides A B and B C, are the respective homologous sides.

Because the polygons Q and R are similar to each other [*Hyp.*], Q is to R as the square of A B is to that of B C; hence, Q and R are together to the sum of the squares of A B and B C, as R is to the square of B C [xv]; therefore, Q and R are together to R, as the squares of A B and B C, are together to the square of B C [xiii]: but the squares of A B and B C, are together equivalent to the square of A C; therefore, Q and R are to R, as the square of A C is to that of B C. Because P and R are similar to each other [*Hyp.*], P is to R as the square of A C is to that of B C [29]; thence P is to R as Q



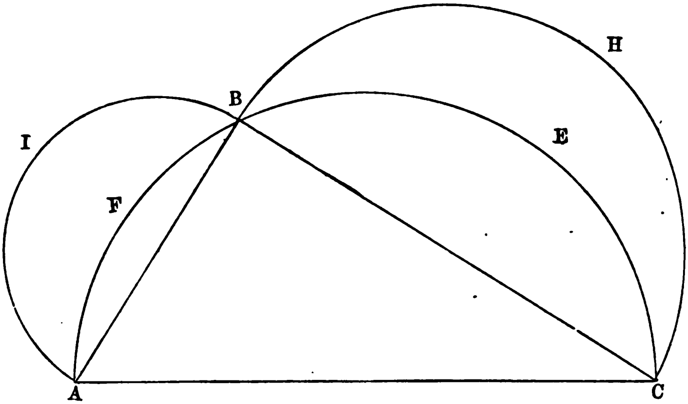
and R together are to R [i]: therefore the polygon P is equivalent to the sum of Q and R [xvi], which W. T. B. D.

COROLLARY. *If the sides of a right-angled triangle be the radii, or diameters, of three circles, or the sides of three similar sectors, or the bases of three similar segments, that upon the hypothenuse is equivalent to the sum of the others.*

THEOREM 37.

If three semicircumferences be described upon the same side of the hypothenuse and the right sides of a right-angled *triangle, the two lunes thus formed are together equivalent to the triangle.*

Let there be a right-angled triangle A B C, and the two lunes A F B I and B E C H formed by three semicircumferences, A F C, A I B and B H C.



Because the semicircles A B I and B C H are similar to each other [III. 82 iii], they are together equivalent to the semicircle A C F [36 i]: but the semicircles A B I and B C H have the segments A B F and B C E in common with the semicircle A C F; therefore the lunes A F B I and B E C H are together equivalent to the triangle A B C [viii], which W. T. B. D.

Corollary. *An isosceles right-angled triangle is equivalent to the lune, formed by the quadrant drawn through the extremities of the base from the apex as a centre, and the semicircumference described on the base.*

Scholium. This proposition gives the means of constructing a curvilinear plane figure equivalent to a rectilinear one.[1]

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[1] This theorem, discovered by Hypocrates, of Chios (about 400 B.C.), is remarkable as showing the only instance when a curvilinear plane figure is exactly equivalent to a rectilinear one; this ingenious discovery, however, has remained barren to this day.

SECTION V.

ON THE MEASUREMENT OF AREAS.

DEFINITIONS.

6. A *square* unit is a square of which each side is equal to the linear unit.

The square unit is generally adopted as the superficial unit for measuring the areas of all plane figures.

7. *Squaring* a plane figure is determining its value.

That is determining the ratio of its area to that of the square unit.

When the length of a straight line bears to the linear unit, the same relation as that which the area of a plane figure bears to the square unit, the straight line and the plane figure have evidently the same value [*Gen. Def.* 15]: the straight line is then called the *measure* of the plane figure. Whence,

8. A straight line is said to be the *measure* of a plane figure when it has the same value.

It is obvious that plane figures which have the same measure are equivalent to each other, for they have then the same value.

9. A regular polygon is said to *roll* upon a straight line when, revolving upon its centre, its sides coincide successively with that line, so that two adjacent sides coincide with a part of the straight line equal to their sum.

Hence, if a polygon rolls upon a straight line, the parts of the latter between the points coinciding successively with the same vertex, are each equal in length to the perimeter.

Evidently, if a circle rolls upon a straight line, the latter is a tangent, for, as the circle is a regular polygon the sides of which are infinitely small, the radii to each side coincide with the apothem; the parts of the straight line between the points of tangence at the same point of the circumference are obviously each equal in length to the circumference.

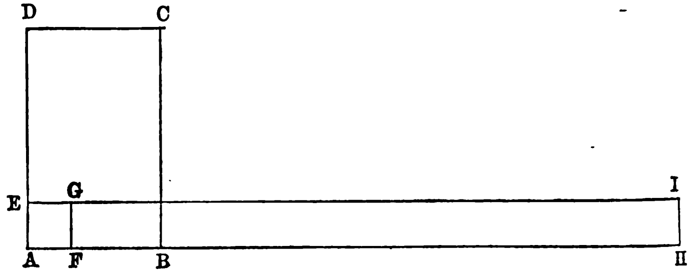
10. A straight line equal in length to a circumference, or an arc, is said to be the circumference, or the arc, *rectified*.[1]

As the sides of the polygon limited by the circumference are infinitely small, that is, so small that it is impossible to realize them in imagination, much more to measure them, it is obvious that it is impossible to find a straight line exactly equal in length to a circumference, or an arc.

THEOREM 38.

The measure of a rectangle is the product of two adjacent sides.

Let there be a rectangle A B C D.



Let any common measure A E to the sides A B and A D be their common unit; let A B be produced to a distance B H equal to as many times A B, as there are units in E D, and let E I and H I be parallel to A H and A E respectively: then A H I E is a rectangle. Because A H contains as many times A B as A D contains A E [*Const.*], the line A H is to A B as A D is to A E [*Gen. Def.* 28]: thence, A H is the product of A B by A D [xviii]; also the rectangles A C and A I are equivalent to each other [20].

Let A F be equal to the unit A E, and let F G be parallel to A E; then A G is a square unit [*Def.* 6]. Because the rectangle A I and the square A G are between the same parallels, A I is to A G as A H is to A F [15]; hence A C is to A G as A H is to A F; that is, the rectangle A C is to a square unit as the product of A B by A D is to a linear unit: thus the value of the rectangle A C is equal to that of the line A H; therefore, the measure of the rectangle A C is the product of A B by A D. The same reasoning could be applied, however small the common measure of A B and A D might be; therefore if A B and A D were incommensurable, the measure of the rectangle A C would still be the product of its adjacent sides, W. W. T. B. D.

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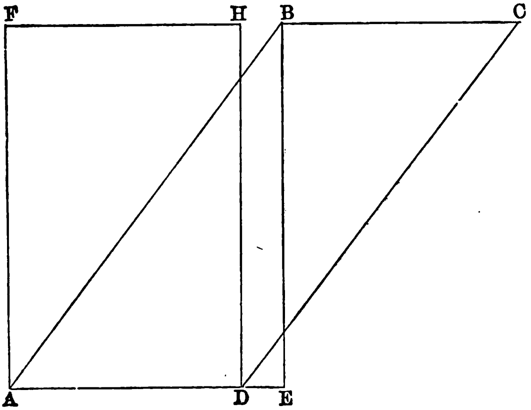
[1] **In the Elements, whenever the length, or the square, or the rectangle, of an arc is spoken of, *that of the same arc rectified* is always meant.**

COROLLARY. *The measure of a square is the product of one of its sides by itself.*

THEOREM 39.

The measure of a parallelogram is the product of its base and altitude.

Let there be a parallelogram A B C D having the base A D and the altitude B E.



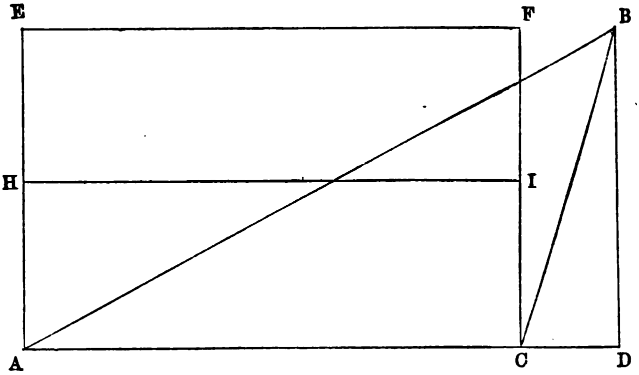
Let A F be equal to B E, and let A F H D be the rectangle of A D and A F; then A H is equivalent to the rectangle of A D by B E. Because the rectangle A H and the parallelogram A C have the same base and equal altitudes, they are equivalent to each other [1]: but the measure of the rectangle A H is the product of A D by A F [38], and A F is equal to B E [*Const.*]; therefore the measure of the parallelogram A C is the product of A D by B E, which W. T. B. D.

COROLLARY. *If the altitude of a parallelogram be equal to its base, it is equivalent to the square of its base.*

THEOREM 40.

The measure of a triangle is the product of its base by half its altitude.

Let there be a triangle A B C, having the base A C, and the altitude B D.



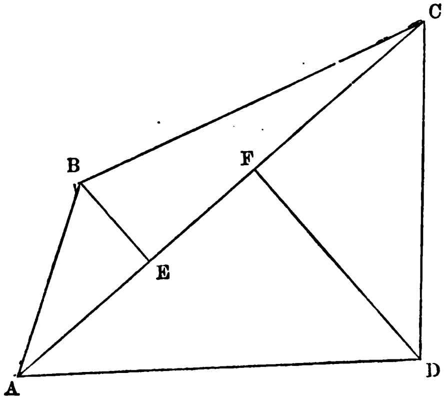
Let A E be equal to B D, and let A E F C be the rectangle of A C by A E; let also H I be the middle line of A E and C F: then the rectangle A I is equivalent to half the rectangle A F. Because the triangle A B C and the rectangle A F have the same base and equal altitude, the triangle A B C is equivalent to half the rectangle A F [3]; thence, the triangle A B C is equivalent to the rectangle A I: but the measure of the rectangle A I is the product of A C by half of A E [38], and A E is equal to B D [*Const.*]; therefore the measure of the triangle A B C is the product of A C by half of B D, which W. T. B. D.

COROLLARY. *The product of the altitude of a triangle by half its base, is the measure of the triangle.*

THEOREM 41.

The measure of a quadrangle is the product of the sum of the altitudes of two opposite vertices upon a diagonal, by half that diagonal.

Let there be a quadrangle A B C D, and let B E, D F be the altitudes of its vertices B and D upon the diagonal A C.



Because the triangle A B C is equivalent to the rectangle of B E by half of A C [3], and the triangle A D C is equivalent to the rectangle of D F by half of A C, the quadrangle A B C D is equivalent to the sum of the rectangles of B E and E F by half of A C: but these rectangles are together equivalent to the rectangle of the sum of B E and D F by half of A C [7]; then the quadrangle A B C D is equivalent to the rectangle of the sum of B E and D F by half of A C; therefore the measure of the quadrangle A B C D is the product of the sum of B E and D F by half of A C [38], which W. T. B. D.

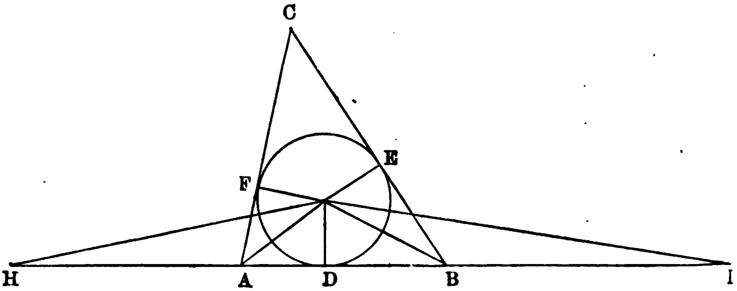
COROLLARY I. *The measure of a lozenge is the product of one diagonal by half the other.*

COROLLARY II. *The product of the altitude of one vertex of a parallelogram upon a diagonal, by that diagonal, is the measure of the parallelogram.*

THEOREM 42.

The product of the perimeter of a triangle by half the radius of the inscribed circle, is the measure of the triangle.

Let the circle O be inscribed in the triangle A B C, and let D, E and F be the points of tangence of the sides.



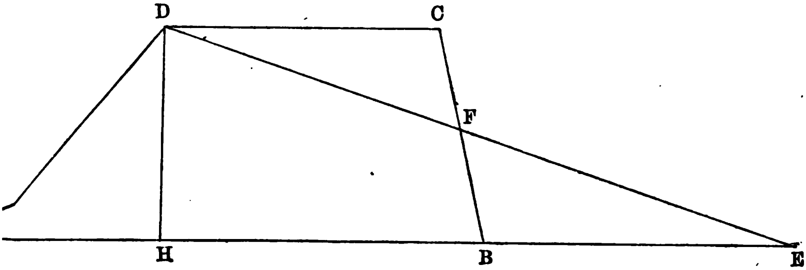
Let the side A B be produced beyond the point A to a distance A H equal to A C, and beyond the point B to a distance B I equal to B C; then H I is equal to the perimeter of the triangle A B C. Let the centre O be joined to the vertices A, B and C, to the points of tangence D, E and F, and also to the points H and I; then the lines O D, O E, and O F, are radii of the circle O. Because the triangles A O C and A O H have equal bases and altitudes, they are equivalent to each other [4]; likewise the triangles B O C and B O I are equivalent to each other: therefore the triangle H O I is equivalent to the triangle A B C. Because O D is perpendicular to A B [I. 53], the product of H I by half of O D is the measure of the triangle H O I; therefore, the product of the perimeter by half the radius O D, is the measure of the triangle A B C, which W. T. B. D.

COROLLARY. *The product of the radius of the circle exscribed to one side of a triangle, by the difference of that side and half the perimeter, is the measure of the triangle.*

THEOREM 43.

The measure of a trapezium is the product of half the sum of its bases by its altitude.

Let A B C D be a trapezium and D H its altitude.



Let A B be produced beyond the point B to a distance B E equal to D C, and let D E be joined, cutting C B in F; then F is the middle point of C B and D E [I. 25 i]. Because the triangles D C F and B E F have the three sides of one respectively equal to the three sides of the other, they are equivalent to each other [III. 17]; therefore the trapezium A B C D is equivalent to the triangle A D E [viii]: but the measure of the triangle A D E is the product of half its base A E by its altitude D H [40], and A E is equal to the sum of A B and D C [*Const.*]; therefore, the measure of the trapezium A B C D, is the product of half the sum of its bases by its altitude, W. W. T. B. D.

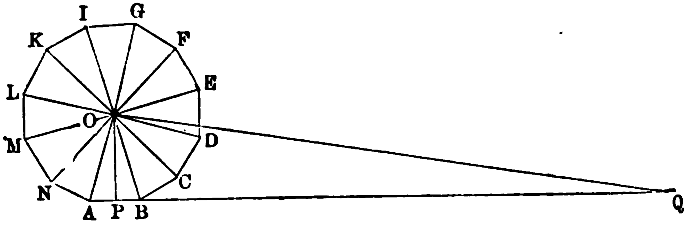
COROLLARY. *The product of the altitude of a trapezium by the middle line of its lateral sides, is the measure of the trapezium.*

THEOREM 44.

The measure of a regular polygon is the product of its perimeter by half its apothem.

Let there be a regular polygon O; let A O B, B O C, C O D, etc., be its sector triangles, and let O P be its apothem.

Let the side A B be produced to a distance B Q equal to the sum of all the others, and let O Q be joined; then A Q is equal to the perimeter of the polygon O, and O P is the altitude of the triangle



A O Q. Because the sector triangles of the polygon O are equivalent to each other [III. 55 i], and there are as many as there are sides, the polygon O is to the triangle A O B as the perimeter is to the side A B [*Gen. Def.* 28]; because the triangles A O Q and A O B have the same altitude, the triangle A O Q is to the triangle A O B as A Q is to A B [15 i]. Now, A Q is equal to the perimeter of the polygon O, then the polygon O is to the triangle A O B as the triangle A O Q is to the triangle A O B [i]; therefore the polygon O is equivalent to the triangle A O Q [xvi]. But the measure of the triangle A O Q is the product of A Q by half of O P [40]; therefore the measure of the polygon O is the product of its perimeter by half of O P [i], which W. T. B. D.

COROLLARY I. *The measure of a circle is the product of its circumference by half its radius.*

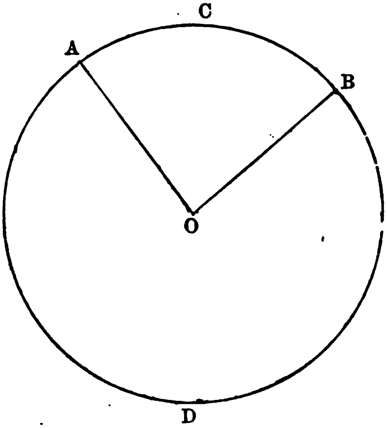
COROLLARY II. *Every circle is equivalent to the rectangle of its circumference by half its radius.*

THEOREM 45.

The measure of a sector is the product of its arc by half one of its sides.

Let there be a sector A O B C.

Let A B D be the circumference of the circle O; then the circle O is a sector the arc of which is the circumference A B D. Because the sector A O B C is a part of the circle O, the sector A O B C is to the circle O as the arc A B is to the circumference A B D [31 ii]. Because the rectangle of the arc A B by half of O A, is to the rectangle of the circumference A B D by half of O A, as the arc A B is to the circumference A B D [15], the sector A O B C is to the circle O, as the rectangle of the arc A B by half of O A, is to that of the circumference A B D by half of O A [i]. But the circle O is equivalent to the rectangle of the circumference A B D by half of O A [44 ii]; then the sector A O B C is equivalent to the rectangle of the arc A B by half of O A: therefore the measure of the sector A O B C, is the product of its arc A B by half its side O A [38], which W. T. B. D.



COROLLARY. *Every sector is equivalent to the rectangle of its arc by half one of its sides.*

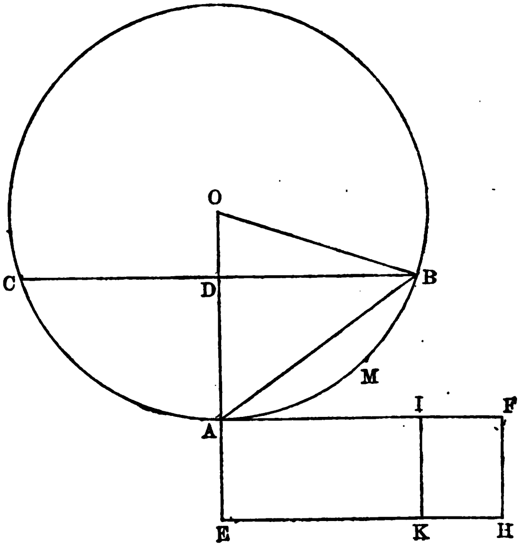
THEOREM 46.

The measure of a segment of a circle, is the product of half the radius, by the difference of the arc and half the chord of twice the arc.

Let there be a segment A B M of the circle O.

Let O B and O A be joined, and let O A be produced to a distance A E, equal to half its length; let A F be a tangent equal to the arc A B rectified, and let A E H F be the rectangle of A E and A F. Let the arc A C be equal to the arc A B, and let B C be joined, cutting

O A in D; then B D is equal to half of B C [I. 44 i]. Let A I be equal to B D, and let I K be a parallel to F H; then A K is equivalent to the rectangle of B D by A E. Because B D is perpendicular to O A [I. 44 i], the triangle O B A is equivalent to the rectangle of B D by half of O A [40 i]; hence, it is equivalent to the rectangle A K. Because the sector A O B M is equivalent to the rectangle A H [45 i], the segment A B M is equivalent to the rectangle I H [ii]; then the measure of the segment A B M is the product of I K by I F: but I K is equal to A E, and I F is equal to the difference between the arc A B and the line B D; therefore, the measure of the segment A B M, is the product of half the radius O A, by the difference of the arc A B and half the chord B C, which W. T. B. D.

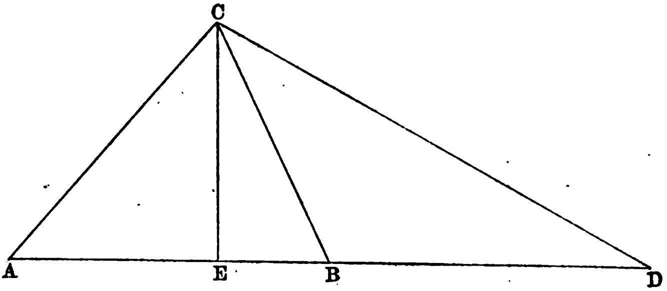


PROBLEMS.

PROBLEM 1.

To construct a triangle equivalent to twice a given triangle.

Let A B C be the given triangle.



Produce A B to a distance B D equal to its length, and join C D: the triangle A D C is the required triangle.

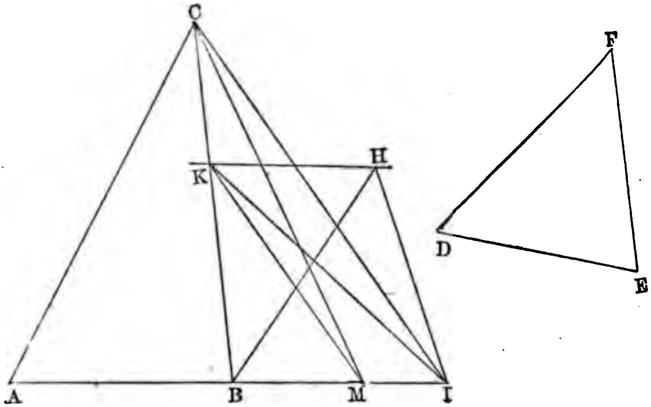
Through the vertex C, draw C E perpendicular to A D [I. *Probl.* 9]; then C E is the altitude of both the triangles A B C and B D C [III. *Def.* 11]. Because the triangles A B C and B D C have equal bases and the same altitude [*Const.*], they are equivalent to each other [4]; therefore the triangle A D C is equivalent to twice the given triangle A B C, which was to be done.

PROBLEM 2. *To construct a triangle equivalent to the product, or quotient, of a given triangle by a given number.*

PROBLEM 3.

To construct a triangle equivalent to the sum of two given triangles.

Let A B C and D E F be the two given triangles.



Produce A B to a distance B I equal to D E, and upon B I construct a triangle B I H equal to the given triangle D E F [III. *Probl.* 1], and join C I; through the vertex H draw H K parallel to I A [I. *Probl.* 10], and cutting B C in a point K; through the point K, draw K M parallel to C I and cutting A I in a point M, and join C M: the triangle A M C is the required triangle.

Join K I. Because the triangles B I H and B I K are between the parallels K H and B I [*Const.*], they are equivalent to each other [4 i]; thence the triangles A B C and B I K are together equivalent to the triangles A B C and B I H [ii]: but the triangles B I H and D E F are equivalent to each other [*Const.*]; therefore the triangles A B C and B I K are together equivalent to the given triangles A B C and D E F. Because the triangles K M I and K M C are between the parallels C I and K M [*Const.*], they are equivalent to each other [4 i]; thence the triangles B M K and K M C are together equivalent to the triangles B M K and K M I [ii]: that is, the triangle B M C is equivalent to the triangle B I K. But the triangles A B C and B I K are together equivalent to the given triangles A B C and D E F

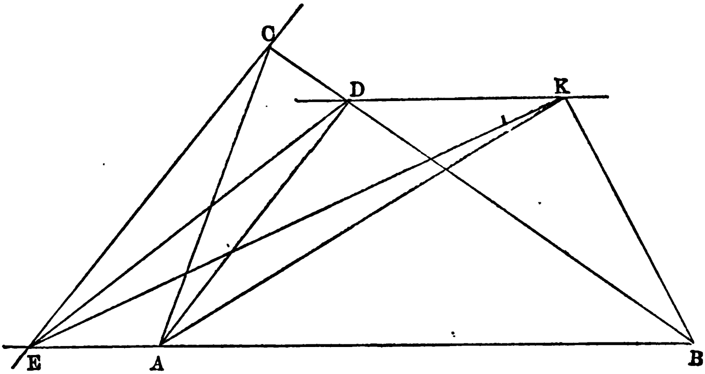
[*Dem.*]; therefore the triangles A B C and B M C are together equivalent to the triangles A B C and D E F [i]: that is, the triangle A M C is equivalent to the sum of the given triangles, W. W. T. B. D.

PROBLEM 4. *To construct a triangle equivalent to the sum of several triangles.*

PROBLEM 5.

To transform a given triangle into another triangle having a given vertex and its base in the same straight line.

Let A B C be the given triangle, and K the given vertex.



Produce the base A B, and, through the given vertex K, draw K D parallel to A B [I. *Probl.* 10] and cutting the side B C in a point D; join D A, and, through the vertex C, draw C E parallel to D A and cutting the prolongation of A B in a point E; join K E and K B: the triangle E B K is the required triangle.

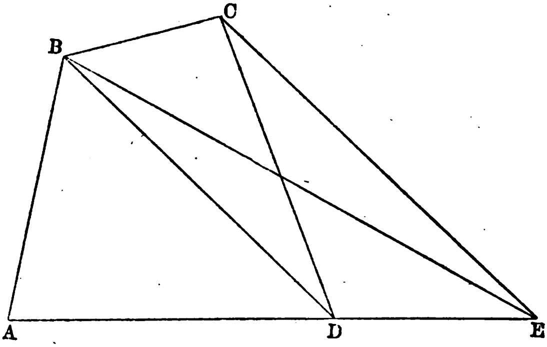
Join A K and E D. Because the triangles A B K and A B D are between the parallels A B and D K [*Const.*], they are equivalent to each other [4 i]. Because the triangles A D E and A D C are between the parallels A D and E C [*Const.*], they are equivalent to each other [4 i]: because the triangles A D E and A E K are between the parallels E B and D K [*Const.*], they are equivalent to each other; hence the triangle A E K is equivalent to the triangle A D C [i]: therefore the triangles A B K and A E K are together equivalent to the triangles A B D and A D C [iv]; that is, the triangle E B K is equivalent to the given triangle A B C, which W. T. B. D.

PROBLEM 6. *To transform a given triangle into another triangle having a given vertex and its base in a given straight line.*

PROBLEM 7.

To transform a given quadrangle into a triangle.

Let A B C D be the given quadrangle.



Produce A D and join B D; through the point C, draw C E parallel to B D [I. *Probl.* 10] and cutting the prolongation of A D in a point E; and join B E: the quadrangle A B C D is now transformed into the triangle A E B.

Because the triangle B D E and B D C are between the parallels B D and C E [*Const.*], they are equivalent to each other [4 i]; hence the triangles A D B and B D E are together equivalent to the triangles A D B and B D C [ii]: therefore the triangle A E B is equivalent to the quadrangle A B C D, which W. T. B. D.

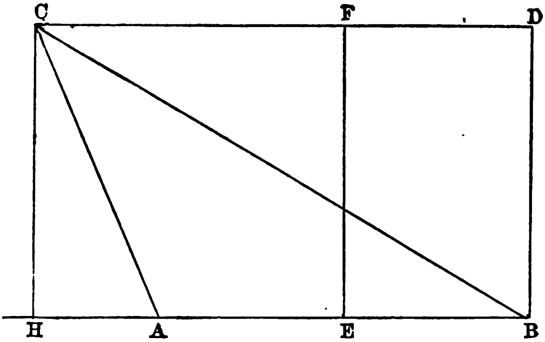
Problem 8. *To transform a given quadrangle, into a triangle having a given vertex, and its base in a given straight line.*

Problem 9. *To transform a given polygon, into a triangle having a given vertex, and its base in a given straight line.*

Problem 10.

To transform a given triangle into a rectangle.

Let A B C be the given triangle.



Through the point B, draw B D perpendicular to A B [I. *Probl.* 5], and, through the vertex C, draw C D parallel to A B [I. *Probl.* 10] and meeting B D in a point D; bisect A B in E [I. *Probl.* 7], and, through the point E, draw E F perpendicular to A B and meeting C D in a point F: the quadrangle E B D F is a rectangle equivalent to the given triangle.

Produce A B, and, from the point C, draw C H perpendicular to A B [I. *Probl.* 9] to meet A B, or its prolongation in H; then C H is the altitude of the given triangle A B C [III. *Def.* 11]. Because C D is parallel to A B [*Const.*], it is perpendicular to B D and E F [I. 17];

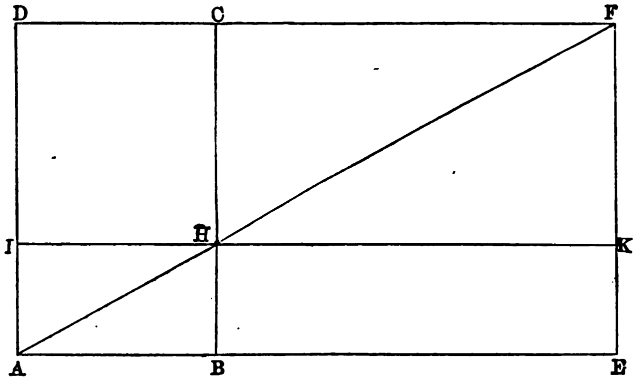
thence, the quadrangle E B D F is a rectangle [III. *Def.* 19]. Because C H is parallel to D B [I. 16], the lines C H and B D are equal to each other [I. 19i]; thence, the rectangle E B D F and the triangle A B C have equal altitudes: but the base E B of the rectangle E B D F is half of the base A B of the triangle A B C [*Const.*]; therefore the rectangle E B D F is equivalent to the given triangle [3 ii], W. W. T. B. D.

PROBLEM 11. *To transform a given triangle into a parallelogram having a given angle* (Eucl. I. 42).

PROBLEM 12.

To transform a given rectangle into another rectangle having a given base.

Let A B C D be the given rectangle, and A E the given base.



Through the point E, draw E F perpendicular to A E [I. *Probl.* 5], and produce D C to meet E F in a point F; join A F, cutting B C in a point H, and, through the point H, draw I K parallel to A E [I. *Probl.* 10] and meeting A D and E F in I and K: the quadrangle A E K I is the required rectangle.

For E D and E I are two rectangles [III. *Def.* 19], and the rectangle E I is equivalent to the given rectangle B D [6 i], which W. T. B. D.

Problem 13. *To transform a given parallelogram into another parallelogram, having a given base and a given angle.*

Problem 14. *To transform a given triangle into a rectangle having a given base, or into a parallelogram having a given base and a given angle.* (Eucl. I. 44.)

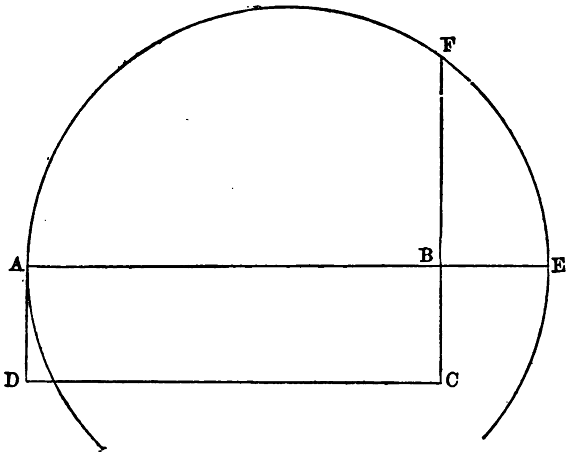
Problem 15. *To transform a given triangle, into another triangle having a given base adjacent to a given angle.*

Problem 16. *To transform a given polygon into a rectangle having a given base, or into a parallelogram having a given base and a given angle.*

Problem 17.

To find a square equivalent to a given rectangle.

Let A B C D be the given rectangle.



Produce A B to a distance B E equal to B C, and upon A E describe a circumference; produce B C to meet the circumference in F: the square of B F is the required square.

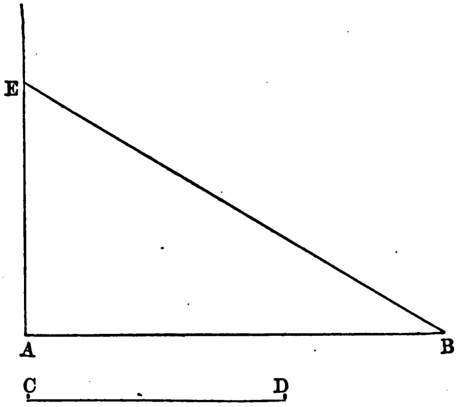
Because B C is perpendicular to A B [*Hyp.*], its prolongation B F is perpendicular to the diameter A E; hence the line B F is a mean proportional to A B and B E [II. 49 ii]; therefore the square of B F is equivalent to the rectangle of A B and B E [20 i], that is, to the given rectangle A C, which W. T. B. D.

PROBLEM 18. *To find a square equivalent to a given polygon.* (Eucl. II. 14.)

PROBLEM 19.

To find a square equivalent to the sum of two given squares.

Let A B and C D be the sides of the given squares.



Through the point A, draw A E perpendicular to A B [I. *Probl.* 5]; take A E equal to C D, and join B E: the square of B E is the required square.

Because the triangle A B E is right-angled in A [*Const.*], the square of B E is equivalent to the sum of the squares of A B and A E [11]: but A E is equal to C D [*Const.*]; then the squares of A E and C D are equivalent to each other [III. 39 i]; therefore the square of B E is equivalent to the sum of the squares of AB and C D, which W. T. B. D.

PROBLEM 20. *To find a square equivalent to twice a given square.*

PROBLEM 21. *To find a square equivalent to the sum of several given squares.*

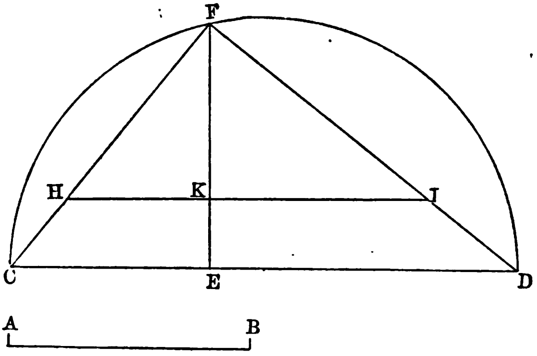
PROBLEM 22. *To find a square equivalent to the sum of two, or any number of, given polygons.*

PROBLEM 23. *To find a square equivalent to the difference of two given polygons.*

PROBLEM 24.

Given a square, to find another square, which shall be to the first in a given ratio.

Let A B be the side of the given square.



Upon a straight line C D, describe a semicircumference C F D, and divide C D in E into the given ratio (II. *Probl.* 8); through the point E, draw E F perpendicular to C D [I. *Probl.* 5] to meet the semicircumference in F; join C F and F D, and take F H equal to A B; through the point H, draw H I parallel to C D [I. *Probl.* 10], meeting F D in a point I, and E F in a point K: the square of F I is the required square.

Because the angle C F D is inscribed in the semicircle C D F [*Const.*], the angle C F D is a right angle [III. 77]; thence, the triangle H F I is a right-angled triangle. Because E F is perpendicular to C D [*Const.*] it is perpendicular to its parallel H I [I. 17]; hence K H and K I are the projections of F H and F I on the hypothenuse H I: therefore the square of F I is to the square F H as K I is to K H [33]. Because H I and C D are parallel to each other [*Const.*], the line E D is to E C as K I is to K H [II. 42]; thence, the square of F I is to the square of F H as E D is to E C [i]: but F I is equal to A B [*Const.*]; then the squares of F I and A B are equivalent to each other [III. 39 i]; therefore, the square of F I is to the square of A B as E D is to E C, that is, in the given ratio, W. W. T. B. D.

PROBLEM 25. *Given a polygon, to construct a similar polygon which shall be to the first in a given ratio.*

PROBLEM 26. *Given a circle, a sector, or a segment, to describe a circle, a similar sector, or segment, which shall be to the first in a given ratio.*

EXERCISES.

I. THEOREMS.

A certain number of exercises are here given as a means of developing the student's ingenuity: they consist in propositions to be proved and problems to be solved.

In demonstrating the propositions or the solutions of the problems, the student may refer to all the theorems and corollaries contained in the four preceding Books. But he should bear in mind that the best demonstrations will always be the shortest; and among these, the preference is to be given to those which refer to the most elementary principles and propositions of Geometry.

For those theorems which require a somewhat more complicated demonstration, the student will find it easier to divide the difficulty by first proving one or more preparatory lemmas, and then dealing with the proposition or theorem itself.

1. The shortest chord which can be drawn through a point within the circumference, is perpendicular to the diameter passing through that point.

2. If an arc is smaller, or greater, than a semicircumference, the part of the diameter intercepted between the arc and its chord is smaller, or greater, than half the chord.

3. The tangent through the middle point of an arc is parallel to the chord, *and conversely.*

4. If three equal chords of a circumference cut one another in the same point, their intersection is the centre.

5. If a circumference be described on the radius of another circumference, each chord of the latter drawn from their common point, is bisected by the former.

6. If several circumferences are tangent to one another in the same point, the radii terminating at the points of section of a common secant through the point of tangence, are parallel to one another.

7. The tangents drawn to two secant circumferences from a point of the prolongation of their common chord, are equal to each other, *and conversely.*

8. A secant is cut by the perpendiculars from the extremities of a diameter, in two points equally distant from the points of section.

9. The measure of an angle supplementary to an inscribed angle, is half the sum of the arcs subtended by the sides of the angle.

10. Two equal obliques to a straight line from a point without it, form with the straight line, two equal angles, *and conversely.*

11. The greater of two obliques to a straight line from a point without it, forms a smaller interior angle with the straight line.

12. A circumscribed angle is double of the angle formed by the chord of the convex arc, with the diameter terminating at one extremity.

13. The difference of two angles inscribed in opposite segments, is equal to the angle circumscribed at the extremities of the common base of the segments.

14. If two equal circumferences be secant to each other, and if from the centre of one there be two secants drawn to the other through the points of section, the concave arc is equal to three times the convex arc.

15. The altitudes of an equilateral triangle are equal to one another.

16. The altitudes of the vertices of the angles at the base of an isosceles triangle, are equal to each other.

17. The bisectrices of the angles at the base of an isosceles triangle, meet the other sides in two points of which the distances from each other and from the extremities of the base, are equal to each other.

18. If one of the angles at the hypothenuse of a right-angled triangle be double of the other, the side opposite the latter is half of that opposite the former.

19. In every triangle the intersections of the altitudes, the medians, and the middle perpendiculars on the sides, are points of the same straight line.

20. Any two sides of a triangle are together greater than twice the median to the third side.

21. Any two angles of an acute-angled triangle are together greater than the third.

22. The angle formed by the bisectrices of the angles at the base of an isosceles triangle, is equal to an exterior angle at the base of the triangle.

23. The bisectrix of an exterior angle at the apex of an isosceles triangle, is parallel to the base, *and conversely.*

24. Two equilateral triangles are equal to each other when they have equal altitudes.

25. Two isosceles triangles are equal to each other when they have equal bases and altitudes.

26. The triangles formed by joining the extremities of two chords which cut each other are similar to each other.

27. Two right-angled triangles are equal to each other:

1. When the perimeter and one right side of the one, are respectively equal to the perimeter and one right side of the other.
2. When the hypothenuse and the altitude of the opposite vertex of the one, are respectively equal to the hypothenuse and the altitude of the opposite vertex of the other.
3. When the perimeter and the hypothenuse of the one, are respectively equal to the perimeter and the hypothenuse of the other.

28. Two triangles are equal to each other:

1. When the base, an adjacent angle, and the altitude of the one, are respectively equal to the base an adjacent angle and the altitude of the other.
2. When one side and the altitudes upon the other sides of one triangle, are respectively equal to one side and the altitudes upon the other sides of the other triangle.
3. When an angle of the one and the altitudes upon its sides, are respectively equal to an angle of the other and the altitudes upon its sides.

4. When they have equal perimeters, and two angles of the one are equal to two angles of the other, each to each.

29. Two triangles inscribed in, or circumscribed to, equal circles are equal to each other:

1. When they are equiangular to each other.
2. When two sides of the one are equal to two sides of the other, each to each.
3. When one side and an adjacent angle of the one, are respectively equal to one side and an adjacent angle of the other.

30. The middle lines of the opposite sides of a quadrangle and that of its diagonals, bisect one another in the same point.

31. The straight lines joining the middle points of opposite sides of a parallelogram to the extremities of one diagonal, divide the other diagonal into three equal parts.

32. The centres of four circles which are each tangent to three sides of a quadrangle, are points of the same circumference.

33. In a quadrangle inscribed in a circle, the straight line joining the feet of the perpendiculars to two opposite sides from the extremities of one diagonal, is parallel to the other diagonal.

34. The exterior angles of a quadrangle made by producing the sides successively, are together equal to the interior angles.

35. The straight line which passes through the intersection of the diagonals of a trapezium and through that of the prolongations of the lateral sides, bisects the bases.

36. *Conversely*, the straight lines which pass through the extremities of the base of a triangle and through a point of the median to the base, cut the other sides on a parallel to the base.

37. Two squares are equal to each other when the difference of the side and diagonal of the one, is equal to that of the side and diagonal of the other.

38. Two parallelograms are equal to each other:

1. When one side and both the diagonals of the one, are respectively equal to one side and the diagonals of the other.
2. When the altitude and both the diagonals of the one, are

respectively equal to the altitude and both the diagonals of the other.

39. Two lozenges are equal to each other:
    1. When the diagonals of the one are equal to those of the other, each to each.
    2. When the perimeter and one angle of the one, are respectively equal to the perimeter and one angle of the other.
40. Two trapeziums are equal to each other:
    1. When the sides of the one are equal to those of the other, each to each.
    2. When one angle, the altitude and bases of the one, are respectively equal to one angle, the altitude and bases of the other.
41. Two quadrangles are equal to each other:
    1. When the sides and one angle of the one, are respectively equal to the sides and one angle of the other.
    2. When two adjacent angles and three sides of the one, are respectively equal to two adjacent angles and three sides of the other.
    3. When three angles and two adjacent sides of the one, are respectively equal to three angles and two adjacent sides of the other.
    4. When the diagonals and three sides of the one are respectively equal to the diagonals and three sides of the other.
42. Two quadrangles inscribed in equal circles are equal to each other:
    1. When the sides of the one are equal to those of the other, each to each.
    2. When two sides and two angles of the one, are equal to two sides and two angles of the other, each to each.
    3. When three sides and one angle of the one, are equal to three sides and one angle of the other, each to each.
    4. When three angles and one side of the one, are equal to three angles and one side of the other, each to each.

43. Two trapeziums are similar to each other:

1. When the sides of the one are proportional to those of the other.
2. When they have an equal angle, and the altitude and bases of the one are proportional to the altitude and bases of the other.

44. Two quadrangles are similar to each other:

1. When, having an equal angle, the sides of the one are proportional, and placed similarly, to those of the other.
2. When two angles of the one are equal to two angles of the other, each to each, and the sides of these angles are proportional.
3. When, having three equal angles, two sides of the one are proportional, and placed similarly, to two sides of the other.

45. The opposite sides of an equiangular hexagon are parallel to each other.

46. The intersections of the diagonals of a regular pentagon are the vertices of a regular pentagon having the same centre.

47. The intersections of the primary diagonals, or the middle points of the sides, of a symmetric or regular polygon, are the vertices of a symmetric or regular polygon having the same axis, or centre.

48. A polygon inscribed in, and circumscribed to, two concentric circumferences, is regular.

49. An equilateral polygon inscribed in a circle is regular.

50. An odd-sided equiangular polygon inscribed in a circle, is regular.

51. If each side of a square be produced to a distance equal to half a diagonal, the extremities of the prolongations are the vertices of a regular octagon having the same centre.

52. If each radius of a regular hexagon be produced to a distance equal to its length, the extremities of the prolongations are the centres of six circles each equal to the circumscribed circle, and each tangent to two others and to the circumscribed circle.

53. In a trapezium, the sum of the squares of the diagonals is equivalent to that of the squares of the lateral sides, together with twice the rectangle of the bases.

54. The sum of the squares of the diagonals of a quadrangle, is equivalent to twice that of the squares of the middle lines of the opposite sides.

55. The sum of the squares of two opposite sides of a quadrangle, is equivalent to twice that of the squares of the middle lines of the other sides, and of the middle line of the diagonals.

II. PROBLEMS.

In solving the following problems, the solution of the problems placed at the end of each of the four preceding Books are supposed to be known.

The student should not forget that the best solutions are the shortest; and among these again the best are those that are obtained by the simplest process of reasoning, and are based upon the most elementary propositions.

For the more complicated problems, the student is recommended to use analysis in trying to solve them; but it is better that he should draw up the solution in a synthetical form.

1. To determine the point of a given straight line, whose sum of distances to two given points is the smallest possible.

2. To draw a perpendicular at the extremity of a given straight line, without producing it.

3. To draw a parallel to a given straight line at a given distance from it.

4. Through a point given without a given straight line, to draw a parallel to that line without describing any arc.

5. To a given circumference, to draw a tangent parallel to a given straight line.

6. To draw a tangent to two given circumferences.

7. To describe a circumference passing through a given point, and tangent to a given straight line in a given point.

8. To describe a circumference in which two chords, given in length, subtend two arcs of which one shall be double of the other.

9. In a given circumference, to draw a chord, given in length, which shall be bisected by a given chord.

10. To describe a circumference which shall have its centre in a given straight line, pass through a given point, and shall be tangent to a given straight line, or circumference.

11. To describe a circumference which shall have its centre in a given straight line, and shall be tangent to a given straight line and circumference, or to two given circumferences.

12. To describe a circumference which shall pass through two given points and be tangent to a given straight line, or which shall pass through a given point and be tangent to two given straight lines, or which shall be tangent to three given straight lines.

13. To describe a circumference which shall pass through two given points and be tangent to a given circumference, or which shall pass through a given point and be tangent to two given circumferences, or which shall be tangent to three given circumferences.

14. To describe a circumference which shall pass through a given point and be tangent to a given straight line and circumference, or which shall be tangent to two given straight lines and to a given circumference, or which shall be tangent to a given straight line and to two given circumferences.

15. With a given radius, to describe a circumference which shall pass through two given points, or which shall be tangent to two given straight lines, or which shall pass through a given point and be tangent to a given straight line.

16. With a given radius, to describe a circumference which shall pass through a given point and be tangent to a given circumference; or which shall be tangent, either to a given straight line and circumference, or to two given circumferences.

17. To describe a circumference which shall pass through a given point and be tangent to a given circumference in a given point, or which shall be tangent to a given straight line and circumference in a point given in the latter or in the former; or which shall be tangent to two given circumferences in a point given in either of them.

18. To a given circumference, to draw a tangent which shall make a given angle with a given straight line, or a secant of which the chord shall have a given length, and which shall make a given angle with a given straight line.

19. Through a point given without a given straight line, to draw another straight line making with it a given angle.

20. Through two given points, to draw two straight lines which shall make a given angle and cut each other in a point of a straight line given in position.

and the radius of the inscribed circle; or the right-angled triangle of which are given the hypothenuse and the radius of the inscribed circle.

28. Given one of the right sides of a right-angled triangle and the sum or difference of the other sides, to construct the triangle.

29. To construct the triangle of which are given:

1. The base, an adjacent angle and the altitude.
2. One side and the altitudes upon the other sides.
3. The altitude and two sides.
4. An angle and the altitudes upon its sides.
5. One side, an adjacent angle and the altitude of its vertex.
6. The altitude and two angles.
7. The perimeter, the altitude and an angle at the base.
8. One side, an adjacent angle, and the sum or difference of the other sides.
9. The base, the altitude and an angle at the base.
10. One side, the opposite angle, and the ratio of the other sides.
11. One side, the opposite angle, and the radius of the inscribed or circumscribed circle.
12. Two angles, and the radius of the inscribed circle.
13. Two sides, and the radius of the inscribed circle.
14. One side, an adjacent angle and the radius of the inscribed circle.
15. Two sides, and the point of tangence of one of them with the inscribed circle.
16. Two sides, and the point of tangence of one of them with the circle exscribed to the third side.
17. One side, the opposite angle and the median from its vertex.
18. One side, or one angle, the radius of the inscribed circle, and the centre line of the inscribed and circumscribed circles.
19. One side, the radius of the circumscribed circle, and the centre line of the inscribed and circumscribed circle.

21. Through a given point, to draw a straight line forming two equal angles with two given straight lines.

22. To trisect a right angle.

23. To divide a given right-angled triangle into two isosceles triangles.

24. To find the locus of:

1. The points equally distant from two given points.
2. The points equally distant from a given straight line, or from two given straight lines.
3. The middle points of all straight lines drawn from a given point to a given straight line.
4. The points equally distant from a given circumference.
5. The middle points of all chords parallel to a chord given in a given circumference.
6. The centres of all circumferences tangent to a given straight line in a given point.
7. The middle points of all chords equal to a chord given in a given circumference.
8. The middle points of all straight lines drawn to a given circumference from a point without it.
9. The centres of all circumferences passing through two given points.
10. The vertices of all angles equal to a given angle, and of which the sides shall pass through two given points.
11. The points of which the distances to two given points are to each other in a given ratio.
12. The vertices of all triangles equivalent to a given triangle upon the same base and on the same side of it.

25. To construct the equilateral triangle of which the altitude is given; or the isosceles triangle of which the altitude and one side, or one angle, are given; or the right-angled triangle of which the hypothenuse and the altitude of the opposite vertex are given.

26. Given the base of an isosceles triangle and the angle at the apex, to construct the triangle.

27. To construct the isosceles triangle of which are given the base

and the radius of the inscribed circle; or the right-angled triangle of which are given the hypothenuse and the radius of the inscribed circle.

28. Given one of the right sides of a right-angled triangle and the sum or difference of the other sides, to construct the triangle.

29. To construct the triangle of which are given:

1. The base, an adjacent angle and the altitude.
2. One side and the altitudes upon the other sides.
3. The altitude and two sides.
4. An angle and the altitudes upon its sides.
5. One side, an adjacent angle and the altitude of its vertex.
6. The altitude and two angles.
7. The perimeter, the altitude and an angle at the base.
8. One side, an adjacent angle, and the sum or difference of the other sides.
9. The base, the altitude and an angle at the base.
10. One side, the opposite angle, and the ratio of the other sides.
11. One side, the opposite angle, and the radius of the inscribed or circumscribed circle.
12. Two angles, and the radius of the inscribed circle.
13. Two sides, and the radius of the inscribed circle.
14. One side, an adjacent angle and the radius of the inscribed circle.
15. Two sides, and the point of tangence of one of them with the inscribed circle.
16. Two sides, and the point of tangence of one of them with the circle exscribed to the third side.
17. One side, the opposite angle and the median from its vertex.
18. One side, or one angle, the radius of the inscribed circle, and the centre line of the inscribed and circumscribed circles.
19. One side, the radius of the circumscribed circle, and [illegible] centre line of the inscribed and circumscribed circle[illegible]

20. One side, the radius of the circle exscribed to another side, and that of the circumscribed circle.
21. One angle, one side and the radius of the circle exscribed to another side.
22. The radius of the circumscribed circle, one side, with the centre line of the circle exscribed to another side and the circumscribed circle.
23. One angle or one side, the radius of the circle exscribed to another side, and the centre line of that circle and the circumscribed circle.
24. The angles, and the distances of the sides to a given point.

30. Upon a given base, to construct an isosceles triangle, of which each of the equal sides shall be double of the base.

31. Upon a given straight line, to construct an isosceles triangle, of which the angle at the apex shall be equal to the half, or treble, of the angles at the base.

32. Through two given points, to draw two straight lines, forming an equilateral triangle with a straight line given in position.

33. To construct a triangle which shall be similar to a given triangle, and have its vertices in three straight lines meeting in the same point.

34. Given the sum, or difference, of the side and diagonal of a square, to construct the square.

35. To construct the rectangle, of which are given:
1. The perimeter, and the angle formed by one side, and a diagonal.
2. One side, and the middle line of two adjacent sides.

36. To construct the parallelogram of which are given:
1. One side and the diagonals.
2. The base, the altitude and one diagonal.
3. The base, the altitude and one angle.

37. To construct the trapezium, of which are given:
1. The sides.
2. The altitudes, the bases and their middle line.

3. The altitude, the bases and a lateral side.
4. An angle, the altitude, one base, and the diagonal opposite the angle.

38. To construct the quadrangle of which are given :

1. The sides, and the middle line of two opposite sides.
2. Two sides, one angle and the diagonals.
3. The sides and one angle.
4. Three sides and two adjacent angles.
5. Three angles and two adjacent sides.
6. The diagonals and three sides.

39. In a given circle to inscribe the quadrangle of which are given :

1. The four sides.
2. Three sides and one angle.
3. Two sides and two angles.
4. One side and three angles.

40. To construct the lozenge of which the diagonals are given.

41. To circumscribe a circle to a given rectangle.

42. To divide a circle into opposite segments, capable of two angles, the one of which shall be equal to five times the other.

43. Through a given point to draw in a given circle a chord, dividing it into segments capable of two angles, the difference of which shall be equal to a given angle.

44. To inscribe, or circumscribe, an equilateral triangle in a given square.

45. To inscribe a square in a given lozenge.

46. To transform an isosceles or scalene triangle into an equilateral triangle.

47. To transform a triangle into a parallelogram of equal perimeter.

48. To transform a given polygon into a triangle having one side and an adjacent angle of the polygon.

49. To bisect a given polygon by a straight line drawn through a given vertex, or through a point given in one of its sides.

50. To divide a given polygon in a given ratio, by a straight line drawn through a given vertex, or through a point given in one of its sides.

51. To divide a given polygon in a given number of equivalent parts, by straight lines drawn through a point given in one of its sides.

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